1. Suppose that $K_1, K_2 < G$ are two normal solvable subgroups of an arbitrary group $G$. Show that

$$K_1K_2 := \{a_1a_2 \mid a_i \in K_i\}$$

is a subgroup of $G$ which is normal and solvable.

2. Show that the alternating group $A_5$ does not contain any subgroups of order 15 and 20.

3. Show that a finite group $G$ is solvable if and only if we can find a series of subgroups

$$G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_k = \{e\},$$

such that $G_i$ is a normal subgroup in $G_{i-1}$, and $G_{i-1}/G_i$ is a cyclic group of prime order.

4. Let $G$ be a non-abelian group of order $p^3$ for some prime $p$. Find the number of conjugacy classes in $G$ and the number of elements in each conjugacy class.

5. Let $G$ be a group and let $H < G$ be a subgroup. The normalizer $\mathcal{N}_G(H)$ of $H$ in $G$ is the stabilizer of the group $H$ under the conjugation action of $G$ on the set of all subgroups on $G$. Compute the normalizers of the following subgroups:

(a) $G = GL_2(\mathbb{R})$, $H$ - the subgroup of all diagonal matrices.
(b) $G = GL_2(\mathbb{R})$, $H$ - the subgroup of all matrices of the form

\[
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}, \quad a \in \mathbb{R}.
\]

6. Let $p$ be a prime number. Show that the group of upper triangular matrices that have diagonal entries equal to 1 is a Sylow $p$-subgroup in $GL_n(\mathbb{Z}/p)$. 

7. Show that every Sylow subgroup in a group of order 100 is abelian.