Math 602. Homework 4  
(due Friday, October 24, 2008)

1. Let $K$ be a group and let $(A, \alpha)$ be a $K$-module. and consider the canonical split extension

$$(o) \quad 0 \to A \to K \rtimes_\alpha A \to K \to 0$$

of $K$ by $(A, \alpha)$. We say that two splittings $s_1, s_2$ of $(o)$ are conjugate if there exists an element $a \in A$ so that for every $x \in K$ we have $a \cdot s_1(x) \cdot a^{-1} = s_2(x)$ in $K \rtimes_\alpha A$.

(a) Given a splitting $s$ of $(o)$ show that the map $d_s : K \to A$ given by $s(x) = (x, d_s(x))$ is a normalized 1-cocycle of $K$ with coefficients in $(A, \alpha)$.

(b) Show that the set of conjugacy classes of splittings of $(o)$ is bijective to $H^1(K, (A, \alpha))$.

2. Let $K$ be a group and let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence of $A$-modules (we will suppress the notation for the action from now on).

(a) For every normalized 1-cocycle $d \in Z^1(K, A'')_0$ construct a group extension $0 \to A' \to G \to K \to 0$ and a commutative diagram

$$
\begin{array}{c}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
\downarrow & & \downarrow d & & \downarrow d & & \downarrow & \\
0 & \longrightarrow & A' & \longrightarrow & G & \longrightarrow & K & \longrightarrow & 0
\end{array}
$$

in which $\tilde{d}$ is a normalized 1-cocycle of $G$ with coefficients in $A$, where the $G$-module structure on $A$ comes from the homomorphism $G \to K$. 

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(b) Show that the construction (a) gives a map

\[ Z^1(K, A'') \rightarrow \text{Ext}(K, A') \]

which fits in the commutative diagram

\[
\begin{array}{ccc}
Z^1(K, A'') & \rightarrow & \text{Ext}(K, A') \\
\downarrow & & \downarrow \\
H^1(K, A'') & \rightarrow & H^2(K, A')
\end{array}
\]

(c) Let \(a : A' \rightarrow A\) and \(b : A \rightarrow A''\) be the maps appearing in the sequence 0 \(\rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0\). Show that the sequence of group homomorphisms

\[ H^1(K, A') \xrightarrow{a \circ (\cdot)} H^1(K, A) \xrightarrow{b \circ (\cdot)} H^1(K, A'') \xrightarrow{\gamma} H^2(K, A') \]

is exact.

3. (a) Show that if \(A\) is the trivial \(K\)-module, then \(H^1(K, A)\) is the group of all homomorphisms from \(K\) to \(A\) and \(H^2(K, A)\) is the group of equivalence classes of central extensions of \(K\) by \(A\).

(b) Let \(K\) be a finite group. Consider the natural short exact sequence of abelian groups

\[
(*) \quad 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0,
\]

and view it as a sequence of trivial \(K\)-modules. Show that the natural map \(\gamma : H^1(K, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(K, \mathbb{Z})\) constructed in the previous problem is an isomorphism, i.e. every central extension of \(K\) by \(\mathbb{Z}\) is obtained by pulling back the extension (*) via a homomorphism \(K \rightarrow \mathbb{Q}/\mathbb{Z}\).