Math 603. Homework 3
(due Monday, February 22, 2010)

1. Let $G$ and $H$ be groups and let $\rho : G \rightarrow GL(V)$, $\lambda : H \rightarrow GL(W)$ be two finite dimensional linear representations over an algebraically closed field. Show that if $\rho$ and $\lambda$ are irreducible, then
\[ \rho \otimes \lambda : G \times H \rightarrow GL(V \otimes W), \quad (\rho \otimes \lambda)(g, h) := \rho(g) \otimes \lambda(h) \]
is also irreducible.

2. Let $K$ be a field of characteristic zero. Let $h(t) \in K[t]$ be a polynomial of degree $n$. Let $A := K[t]/(h)$.

(a) Show that $A$ is semi-simple if and only if $h$ has no multiple roots in $\overline{K}$.

(b) Let $c_1, \ldots, c_n \in \overline{K}$ be all roots of $h$ (counted with multiplicities if necessary). Let $f(t) \in K[t]$, and let $[f] = f(t) + (h)$ be the corresponding element in $A$. Show that
\[ \text{tr} (T_{[f]}) = \sum_{i} f(c_i). \]

(c) Let $s_i := c_1^i + \ldots + c_n^i$ be the power sums of the roots of $h$. Note that all the $s_i$ can be expressed as polynomials in the coefficients of $h$ and so are in $K$.

Use the scalar product criterion for semi-simplicity to argue that the polynomial $h$ has no multiple roots in $\overline{K}$ if and only if the matrix
\[
\begin{pmatrix}
  s_0 & s_1 & \cdots & s_{n-1} \\
  s_1 & s_2 & \cdots & s_n \\
  \cdots & \cdots & \cdots & \cdots \\
  s_{n-1} & s_n & \cdots & s_{2n-1}
\end{pmatrix}
\]
is non-degenerate.

3. Let $K$ be any field. Show that every commutative finite dimensional simple algebra $A$ over $K$ is either a field extension of $K$ or is the one dimensional algebra with zero multiplication.

4. Let $A$ be a finite dimensional semi-simple associative $\mathbb{C}$-algebra. Let $(W, \lambda)$ be a finite dimensional complex irreducible representation of $A$ of dimension $n$. Consider the representation of $A$ given by $(V, \rho) := (W^{\oplus m}, \rho^{\oplus m})$. What is the group of automorphisms of $(V, \rho)$?