1. Let $L \supset K$ be an extension of fields of characteristic $p > 0$. An element $\alpha \in L$ is separable over $K$ if $\alpha$ is algebraic over $K$ and the minimal polynomial of $\alpha$ is separable over $K$. The separable closure $K^{\text{sep}}$ of $K$ in $L$ is the set of all elements of $L$ that are separable over $K$.

(a) Show that $K^{\text{sep}}$ is a subfield in $L$.

(b) If $L \supset K$ is finite, show that there exists an integer $k \geq 0$ so that $L^{p^k} \subset K^{\text{sep}}$.

(c) If $L \supset K$ is algebraic, show that for any $\alpha \in L$ there exists an integer $k \geq 0$ so that $\alpha^{p^k} \in K^{\text{sep}}$.

2. A finite extension $L \supset K$ of fields is called purely inseparable if $K^{\text{sep}} = K$. Suppose that we have a tower $K = K_0 \subset K_1 \subset \ldots \subset K_s = L$. Show that $L$ is purely inseparable over $K$ if and only if $K_i$ is purely inseparable over $K_{i-1}$ for all $i = 1, \ldots, s$.

3. Show that $\mathbb{Q}(\sqrt{2} + \sqrt{2})$ is a Galois extension of $\mathbb{Q}$ of degree four. Show that the Galois group $\text{Gal}(\mathbb{Q}(\sqrt{2} + \sqrt{2})|\mathbb{Q})$ is cyclic.