1. Let $H$ be the three dimensional complex Heisenberg group, i.e. the group

$$H = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C} \right\}.$$ 

Describe the Lie algebra $\mathfrak{h}$ of $H$ as an algebra of $3 \times 3$ matrices and show that the matrix exponential map \( \exp : \mathfrak{h} \to H \) is a biholomorphism.

2. Let $\mathbb{P}^n$ be a complex projective space with homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$. Let $f_1, \ldots, f_k$ be homogeneous polynomials in the $x_i$'s and let $V = V(f_1, \ldots, f_k)$ be the projective variety they define. Consider the Jacobian matrix $J(f_1, \ldots, f_k)$.

(a) Show that the locus of all points $x \in \mathbb{P}^n$ where $J(f_1, \ldots, f_k)$ has rank $\leq s$ is a projective subvariety. Denote this variety by $j^s V$.

(b) Prove that there exists a minimal integer $m$ such that $j^m V \cap V = V$. If $j^{m-1} V \cap V = \emptyset$ show that $V$ is a compact complex manifold of dimension $n - m$.

3. Consider the action of $\mathbb{Z}$ on $\mathbb{C}^2 - \{0\}$ given by $k \cdot (z_1, z_2) := (2^{-k}z_1, 2^{-k}z_2)$ and let $X := (\mathbb{C}^2 - \{0\})/\mathbb{Z}$ be the corresponding Hopf surface. Let $\pi : X \to \mathbb{P}^1$ be the natural holomorphic map with elliptic fibers. Show that if $Z \subset X$ is a compact connected complex submanifold of positive dimension, then either $Z = X$ or $Z$ is equal to a fiber of $\pi$.

4. Fix a complex number $\tau$ with $\text{Im}(\tau) > 0$. Consider the holomorphic action of the group $G = (\mathbb{C}, +)$ on the complex manifold $\mathbb{C}^\times \times \mathbb{C}^\times$ given by

$$z \cdot (u_1, u_2) := (e^{2\pi iz} u_1, e^{2\pi i\tau z} u_2).$$

(a) Show that $G$ acts freely on $\mathbb{C}^\times \times \mathbb{C}^\times$.

(b) Show that the quotient $X := (\mathbb{C}^\times \times \mathbb{C}^\times)/G$ has a natural structure of an elliptic curve, so that the projection map $\mathbb{C}^\times \times \mathbb{C}^\times \to X$ is holomorphic.

(c) Is $\mathbb{C}^\times \times \mathbb{C}^\times \to X$ a holomorphic line bundle?