1. Let $X$ be a compact Kähler manifold. By the Kähler identities, the Lefschetz operators for any fixed Kähler metric commute with the corresponding Laplacian and hence descend to an $\mathfrak{sl}_2$ triple on the cohomology $H^•_{dR}(X, \mathbb{C})$.

(a) Let $\omega$ be a Kähler form on $X$. Show that the associated Lefschetz operator $L : H^k_{dR}(X, \mathbb{C}) \to H^{k+2}_{dR}(X, \mathbb{C})$ depends only on the cohomology class $[\omega] \in H^2(X, \mathbb{C})$.

(b) Let $\omega'$ be another Kähler form on $X$ which is cohomologous to $\omega$. Let $\{L, \Lambda, D\} \in \text{End}(H^k_{dR}(X, \mathbb{C}))$ and $\{L, \Lambda', D\} \in \text{End}(H^k_{dR}(X, \mathbb{C}))$ be the $\mathfrak{sl}_2$ triples corresponding to $\omega$ and $\omega'$. Use the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ to argue that $\Lambda = \Lambda'$.

2. Use the Hodge theorem and the existence of the Green operator to prove the principle of the two types: If $X$ is compact Kähler and if $\alpha \in A^{p,q}(X)$ is a differential form satisfying $\partial \alpha = 0$ and $\bar{\partial} \alpha = 0$ and is either $\partial$- or $\bar{\partial}$-exact, then there exists a form $\gamma \in A^{p-1,q-1}(X)$ with $\partial \bar{\partial} \gamma = \alpha$.

3. Let $f : X \to Y$ be a surjective holomorphic map between compact complex manifolds.

(a) Show that if $X$ is Kähler, then for any $k$ the induced map on cohomology $f^* : H^k(Y, \mathbb{C}) \to H^k(X, \mathbb{C})$ is injective.

(b) Let $X$ be a Hopf surface, $Y = \mathbb{P}^1$ and $f : X \to Y$ the natural map. Is the map $f^*$ injective?

4. Let $X$ be a compact Kähler manifold of dimension $n$. Let $\epsilon : \hat{X} \to X$ be the blow-up of $X$ at a point. Show that for every $k$ we have

$$H^k(\hat{X}, \mathbb{Z}) \cong \begin{cases} H^k(X, \mathbb{Z}) \oplus \mathbb{Z}(k/2), & \text{for } k \neq 0 \text{ even} \\ H^k(X, \mathbb{Z}), & \text{otherwise} \end{cases}$$

as Hodge structures.