1 Minkowski space

An $n$-dimensional *Minkowski spacetime* is an $n$ dimensional real vector space $\mathbb{M}^n$ equipped with a non-degenerate symmetric scalar product $\eta$ of signature $(1, n-1)$.

The concrete model for $(\mathbb{M}^n, \eta)$, that one usually takes, consists of $n$-dimensional coordinate space $\mathbb{R}^n$ (with coordinates $x^0, x^1, \ldots, x^{n-1}$) and the quadratic form:

$$\eta(x) := (x^0)^2 - (x^1)^2 - \ldots - (x^{n-1})^2.$$ 

In the standard basis he Gram matrix of $\eta$ is

$$\eta = (\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{pmatrix}$$

The following sequence of exercises explores the special properties of the 4 dimensional Minkowski space:

1.1. Fix a two dimensional complex vector space $\mathbf{H}$. Let $\mathbb{M}^4$ be the space of all (not necessarily non-degenerate) hermitian scalar products $l$ on $\mathbf{H}$. Consider the quadratic form

$$\eta : \mathbb{M}^4 \to \mathbb{R}, \quad \eta(l) := \det(\text{Gram}_l),$$

assigning to each hermitian scalar product $l$ the determinant of its Gram matrix. Show that
Any choice of a basis \( \{ h_1, h_2 \} \) of \( H \) identifies \( \mathbb{M}^4 \) with the space of all \( 2 \times 2 \) hermitian matrices and the quadratic function \( \eta \) with the determinant function.

The Pauli matrices
\[
\begin{align*}
\sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\end{align*}
\]
form a basis for the space of \( 2 \times 2 \) hermitian matrices and argue that \( \mathbb{M}^4 \) is a 4-dimensional real vector space.

If \( \{ h_1, h_2 \} \) is a fixed basis of \( H \) and if

\[
\text{Gram} : \quad \mathbb{M}^4 \longrightarrow \{(2 \times 2) \text{ Hermitian matrices }\}
\]

is the map assigning to each hermitian product \( l \) its Gram matrix \( \text{Gram}_l \) in the basis \( \{ h_1, h_2 \} \), then the bilinear form \((\cdot, \cdot)\) polarizing the quadratic form \( \eta \) is given by:

\[
(l_1, l_2) = \frac{1}{2} (\text{tr}(\text{Gram}_{l_1}) \text{tr}(\text{Gram}_{l_2}) - \text{tr}(\text{Gram}_{l_1} \text{Gram}_{l_2})).
\]

Use (iii) to argue that the Pauli matrices correspond to an \( \eta \)-orthonormal basis of \( \mathbb{M}^4 \) and that \( \eta \) has signature \((1, 3)\).

1.2. Use problem 1.1 (i) to identify \((\mathbb{M}^4, \eta)\) with the space of \( 2 \times 2 \) hermitian matrices with the determinant form. Show that the isometry

\[
(\mathbb{M}^4, \eta) \cong (\mathbb{R}^4, \text{diag}(-1, 1, 1, 1))
\]

given by the choice of the Pauli matrices as a basis is explicitly given by

\[
X \mapsto \left\{ x^\mu := \frac{1}{2} \text{tr}(X \sigma_\mu) \right\}.
\]

1.3. Let \( \{ e_\mu \}_\mu \) and \( \{ e'_\mu \}_\mu \) be two \( \eta \)-orthonormal bases of \( \mathbb{M}^4 \). As usual we declare \( \{ e_\mu \} \) and \( \{ e'_\mu \} \) to have the same orientation if we can find a continuous family \( f_t : \mathbb{M}^4 \rightarrow \mathbb{M}^4 \) of isometries of \( \eta \) parameterized by \( t \in [0, 1] \) and such that \( f_0 = \text{id}, f_1(e_\mu) = e'_\mu \). Show that if two bases \( \{ e_\mu \} \) and \( \{ e'_\mu \} \) have the same orientation then
(a) \((e_0, e'_0) > 0;\)
(b) if \(A\) is the matrix of the orthogonal projection map
\[ \bigoplus_{\mu=1}^{3} \mathbb{R} e_\mu \to \bigoplus_{\mu=1}^{3} \mathbb{R} e'_\mu \]
written in the bases \(\{e_\mu\}\) and \(\{e'_\mu\}\), then \(\det(A) > 0.\)

One says that two \(\eta\)-orthonormal bases satisfying (a) have the same
\textit{temporal orientation} and two bases satisfying (b) have the same \textit{spatial orientation}.

1.4. Show that two \(\eta\)-orthonormal bases of \(\mathbb{M}^4\) that have the same
temporal and spatial orientation must have the same orientation. Generalize this
statement to the case of the \(n\)-dimensional Minkowski space \((\mathbb{M}^n, \eta)\).

2 The proper Lorentz group and its universal cover

The \textit{Lorentz group} in \(n\) dimensions is the group \(L\) of isometries of the \(n\)-dimensional Minkowski space \((\mathbb{M}^n, \eta)\). If we choose an \(\eta\)-orthonormal basis of \(\mathbb{M}^n\) we can identify \(L\) with the orthogonal group \(O(1, n-1)\). Problems 1.3 and 1.4 show that \(L\) has four connected components. We will denote these by \(L^+\) (all elements in \(L\) preserving the orientation of some orthonormal basis), \(L^-\) (all elements in \(L\) preserving the temporal but reversing the spatial orientation of an orthonormal basis), \(L^\perp^+\) (all elements in \(L\) reversing the temporal but preserving the spatial orientation of an orthonormal basis), and \(L^\perp^-\) (all elements in \(L\) reversing the temporal and the spatial orientation of an orthonormal basis) respectively.

Note that \(L^\perp^+ \subset L\) is a subgroup and is the connected component of \(L\) containing the identity element. Sometimes the group \(L^\perp^+\) is called the \textit{proper Lorentz group} in dimension \(n\).

2.1. Realize the Minkowski space \((\mathbb{M}^4, \eta)\) as the space of all \(2 \times 2\) hermitian
matrices equipped with the determinant form. Show that the natural map
\[ s : \quad SL(2, \mathbb{C}) \longrightarrow GL(\mathbb{M}^4) \]
\[ A \longrightarrow (X \mapsto AXA^\dagger) \]
gives a surjective group homomorphism from $SL(2, \mathbb{C})$ to $L^+ \subset GL(M^4)$. Show that $\text{ker}(s) = \{ \pm I_2 \}$.

3 Spinors and chirality

Let $W$ be a finite dimensional vector space over $k$ ($= \mathbb{R}$ or $\mathbb{C}$), and let $g$ be a symmetric $k$-valued scalar product on $W$. Recall that the Clifford algebra $\text{Cliff}(W, g)$ is the unique up to isomorphism algebra which admits a $k$-linear inclusion

$$\rho : W \rightarrow \text{Cliff}(W, g)$$

with the properties

- For all $x \in W$ we have $\rho(x) = g(x, x) \cdot 1$;
- $\rho(W)$ generates $\text{Cliff}(W, g)$ as an algebra.

The Clifford algebra $\text{Cliff}(W, g)$ has the following basic property (which actually determines it up to isomorphism):

Suppose that $A$ is any $k$ algebra with unit and let $\tau : W \rightarrow A$ be any $k$-linear map for which

$$\tau(x)^2 = g(x, x) \cdot 1.$$

Then there is a unique extension of $\tau$ to an algebra homomorphism from $\text{Cliff}(W, g)$ to $A$. Note that this property implies that if $W$ is a real vector space, then $\text{Cliff}(W \otimes \mathbb{C}, g \otimes \mathbb{C}) = \text{Cliff}(W, g) \otimes \mathbb{C}$. In particular:

Any complex representation of $\text{Cliff}(W, g)$ is a representation of $\text{Cliff}(W \otimes \mathbb{C}, g \otimes \mathbb{C})$. ($\ast$)

3.1. Consider the Dirac matrices $\gamma_\mu \in GL(4, \mathbb{C})$, $\mu = 0, 1, 2, 3$ defined as the block matrices

$$\gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ -\sigma_\mu & 0 \end{pmatrix}, \mu = 1, 2, 3.$$

Show that $\gamma_\mu$ obey the relations:

$$\gamma_0^2 = -\gamma_1^2 = -\gamma_2^2 = -\gamma_3^2,$$

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 0, \text{ for } \mu \neq \nu.$$
3.2. Let \((M^4, \eta)\) be the 4-dimensional Minkowski space, realized as the space of all \(2 \times 2\) hermitian matrices. Let 

\[ M^4 \otimes \mathbb{C} \rightarrow \text{Mat}_{4\times 4}(\mathbb{C}) \]

be the \(\mathbb{C}\)-linear map sending \(\sigma_\mu\) to \(\gamma_\mu\). Show that this map induces an isomorphism of algebras:

\[ \text{Cliff}\left(M^4 \otimes \mathbb{C}, \eta \otimes \mathbb{C}\right) \rightarrow \text{Mat}_{4\times 4}(\mathbb{C}). \]

**Definition** Let \((W, g)\) be a real vector space equipped with a symmetric scalar product. The space of Dirac spinors for \((W, g)\) is the smallest complex representation of \(\text{Cliff}(W, g)\).

Problem 3.2 implies that the space of Dirac spinors for the Minkowski space \((M^4, \eta)\) is a 4-dimensional complex vector space \(S \cong \mathbb{C}^4\).

If \((W, g)\) is a real space with a non-degenerate scalar product, then the spin group of \((W, g)\) is defined as the universal cover \(\text{Spin}(W, g)\) of the connected component of the group \(O(W, g)\) of \(g\)-orthogonal transformations of \(W\). In the case of the \(n\)-dimensional Minkowski space, the map \(\text{Spin}(M^n, \eta) \rightarrow L^+_1\) is known to be two-to-one. According to problem 2.1 for the 4-dimensional Minkowski space one gets that \(\text{Spin}(M^4, \eta) \cong SL(2, \mathbb{C})\) as real groups.

An important feature of the Clifford algebra is that \(\text{Spin}(M^n, \eta)\) embeds in the group \(\text{Cliff}(M^n, \eta)^\times\) of invertible elements of the Clifford algebra:

4.1. Given any two vectors \(x, y \in M^n\) consider the element 

\[ a = \frac{1}{2}(x \cdot y - y \cdot x) \in \text{Cliff}(M^n, \eta) \]

and the linear operator 

\[ \phi_a : \text{Cliff}(M^n, \eta) \rightarrow \text{Cliff}(M^n, \eta), \quad \xi \mapsto a \cdot \xi - \xi \cdot a. \]

(i) Show that \(\phi_a\) leaves \(M^n\) invariant.

(ii) Show that the linear subspace \(\text{spin}(M^n, \eta)\) in the Clifford algebra, spanned by the elements \(a\) is a Lie subalgebra.
(iii) Show that the map
\[ \phi : \text{spin}(\mathbb{M}^n, \eta) \to \text{so}(\mathbb{M}^n, \eta) \]

is an isomorphism of Lie algebras.

4.2 Define the group Spin(\mathbb{M}^n, \eta) as the connected Lie subgroup of Cliff(\mathbb{M}^4, \eta) with Lie algebra equal to spin(\mathbb{M}^n, \eta).

(i) Show that the homomorphism
\[ \Phi : \text{Spin}(\mathbb{M}^n, \eta) \to GL(\text{Cliff}(\mathbb{M}^n, \eta)), \quad \gamma \mapsto (\xi \mapsto \gamma \cdot \xi \cdot \gamma^{-1}) \]

preserves \( \mathbb{M}^n \) and that for each \( \gamma \), \( \Phi_\gamma \) is an \( \eta \)-orthogonal transformation.

(ii) Show that \( \phi \) is the tangent map at the identity to \( \Phi \).

(iii) Show that \(-1 \in \ker(\Phi)\) and so \( \Phi \) can not be an isomorphism. Argue that \( \Phi \) is the universal cover map for \( \mathbf{L}_+ \).

In view of the previous problem we conclude:

Any representation of Cliff(\mathbb{M}^n, \eta) restricts to a representation of Spin(\mathbb{M}^n, \eta). \hfill \text{(**)}

In particular \((*)\) and \((**)\) imply that \(SL(2, \mathbb{C})\) (viewed as a real group) acts on the space \( S \) of Dirac spinors.

4.3. (i) Show that the action of \( SL(2, \mathbb{C}) \) on \( S \) commutes with the action of the element \( \Gamma = i\gamma_0\gamma_1\gamma_2\gamma_3 \in \text{Cliff}(\mathbb{M}^4 \otimes \mathbb{C}, \eta \otimes \mathbb{C}) \).

(ii) Show that \( \Gamma^2 = I_4 \) and let \( S_L \subset S \) and \( S_R \subset S \) be the \( \pm 1 \) eigenspaces of \( \Gamma \). The space \( S_L \) (respectively \( S_R \)) is called the space of left (respectively right) chirality Dirac spinors. Argue that \( S_L \) and \( S_R \) are two dimensional complex representations of the real group \( SL(2, \mathbb{C}) \).

(iii) Show that \( S_L \) is isomorphic to the fundamental representation of the complex group \( SL(2, \mathbb{C}) \) and that \( S_R \) is isomorphic to the complex conjugate of the fundamental representation.
4 The Dirac equation

Let $\mathcal{C}$ be the space of all $C^\infty$-maps $\psi : M^4 \to S$. Given a map $\psi$ define its Dirac conjugate $\bar{\psi}$ by the formula

$$\bar{\psi} := \psi^\dagger \gamma_0,$$

where $\psi \mapsto \psi^\dagger$ is understood as the chirality changing operator.

5.1 Fix a real constant $m$. Show that the Lagrangian

$$\mathcal{L} : \mathcal{C} \to \mathbb{R}, \quad \psi(x) \mapsto \int dx \bar{\psi}(x) \left( i \sum_{\mu=0}^{3} \gamma_\mu \partial_\mu - m \right) \psi(x)$$

is invariant under the action of $SL(2, \mathbb{C})$.

The corresponding Euler-Lagrange equation reads

$$\left( i \sum_{\mu=0}^{3} \gamma_\mu \partial_\mu - m \right) \psi(x) = 0$$

and is called the Dirac equation. The differential operator $i \sum_{\mu=0}^{3} \gamma_\mu \partial_\mu - m \cdot I_4$ is the physicist's Dirac operator.

In terms of the left and right handed spinors the Dirac equation can be written as

$$\begin{pmatrix}
    i \left( \sigma_0 \partial_0 - \sum_{\mu=1}^{3} \sigma_\mu \partial_\mu \right) \psi_R \\
    i \left( \sigma_0 \partial_0 + \sum_{\mu=1}^{3} \sigma_\mu \partial_\mu \right) \psi_L
\end{pmatrix} = \begin{pmatrix}
    m \psi_L \\
    m \psi_R
\end{pmatrix}.$$

Note that when $m = 0$ one can write a $SL(2, \mathbb{C})$-invariant equation for each spinor separately, but when $m \neq 0$ Lorentz invariance necessarily couples the two equations.

Finally we have an invariance with respect to global $U(1)$ gauge symmetries:

5.2. Show that the $U(1)$-action $\psi \mapsto e^{i\theta} \psi$ preserves the Lagrangian $\mathcal{L}$ and the Dirac equation. Show that the conserved currents for this $U(1)$ action are the functions $j_\mu : \mathcal{C} \to C^\infty(M^4)$, $\psi \mapsto \bar{\psi} \gamma_\mu \psi$. 

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