Higgs bundles and fundamental group schemes

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(relying on joint work with D. Hernández Ruipérez, B. Graña Otero, V. Lanza, A. Lo Giudice, ...)

Definition

A neutral Tannakian category *over a field* $\Bbbk$ *is a rigid Abelian* $\Bbbk$*-linear tensor category* $\mathfrak{T}$ *together with an exact faithful* $\Bbbk$*-linear tensor functor* $\omega\colon \mathfrak{T} \longrightarrow \mathbf{Vect}_{\Bbbk}$*, called the* fiber functor.

"Rigid" means that

- Hom and $\otimes$ satisfy a distributive property (over finite families);
- all objects in $\mathfrak{T}$ are reflexive (i.e., the natural map to their double dual is an isomorphism).

Archetypical Tannakian category: Rep(G) for G an affine group scheme over $\mathbb{k}$

$$\omega(\rho, V) = V \qquad \text{if} \qquad \rho\colon G \to \mathrm{Aut}(V).$$

Theorem (Tannaka duality)

For every neutral Tannakian category $(\mathfrak{T}, \omega)$ there is an affine group scheme G such that $\mathfrak{T} \simeq \mathrm{Rep}(G)$.

Actually $G = \mathrm{Aut}^{\otimes}(\omega)$.

Nori fundamental group

A vector bundle E over a scheme X is essentialy finite if there exists a principal bundle $\pi\colon P \to X$, with a finite structure group, such that π^*E is trivial.

Essentially finite vector bundles make up a neutral Tannakian category (the fiber functor maps E to the fiber over a fixed point $x \in X$). The affine group scheme representing this Tannakian category is the Nori fundamental group scheme $\pi_1^N(X, x)$.

Relation with the usual fundamental group:

$$\boxed{\pi_1^N(X,x) \twoheadrightarrow \pi_1^{\text{ét}}(X,x)}$$

$\pi_1^{\text{ét}}(X,x) = \varprojlim_i \operatorname{Aut}((X_i, x_i))$

((X_i, x_i) pointed Galois coverings of (X, x))

$\pi_1^{\text{ét}}(X,x)$ is a pro-finite completion of $\pi_1(X,x)$

Pro-finite completion of a group G:

$\hat{G} = \varprojlim_i G/N_i$

(N_i normal subgroups of finite index)

Higgs geometry

X a smooth projective variety

A Higgs bundle $\mathfrak{E}$ is a pair (E, ϕ), where E is a vector bundle, and

$$\phi \colon E \to E \otimes \Omega^1_X$$

(the Higgs field) is a morphism such that $\phi \wedge \phi = 0$, where

$$\phi \wedge \phi \colon E \to E \otimes \Omega^2_X$$

The existence of properly semistable Higgs bundles on a variety (or Higgs bundles satisfying some other specific property) is a geometric feature of the variety

(cf. the derived category of coherent sheaves on X)

The classical theory

I will write formulas assuming that X a smooth projective variety over $\mathbb{C}$. But this actually works on any field $\mathbb{k}$ of characteristic zero.

$H = c_1(\mathcal{O}_X(1))$

A line bundle L on X is numerically effective (nef) if for every morphism $f\colon C \to X$, where C is a smooth projective irreducible curve, one has

$$\deg f^*L = \int_C f^*c_1(L) \geq 0.$$

If E is a vector bundle on X, the projectivization $\mathbb{P}E$ carries a "relative hyperplane bundle" $\mathcal{O}_{\mathbb{P}E}(1)$. The vector bundle E is said to be nef if $\mathcal{O}_{\mathbb{P}E}(1)$ is.

Definition of $\mathbb{P}E$:

$$E \quad \rightsquigarrow \quad \mathrm{GL}(E) \quad \rightsquigarrow \quad \mathrm{PGL}(E) \quad \rightsquigarrow \quad \mathbb{P}E$$

E is said to be numerically flat (nflat) if both E and E^* are nef.

Theorem (Demailly-Peternell-Schneider 1994)

A vector bundle E is nflat if and only if it admits a filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

whose quotients E_k/E_{k-1} are Hermitian flat vector bundles (i.e., they are given by representations $\pi_1(X) \to U(r_k)$).

Consequence: all Chern classes of an nflat bundle vanish (as E and $\bigoplus_k E_k/E_{k-1}$ have the same Chern classes)

Stability

Slope of a coherent sheaf $\mathcal{F}$ of positive rank:

$$\mu(\mathcal{F}) = \frac{c_1(\mathcal{F}) \cdot H^{n-1}}{\operatorname{rk} \mathcal{F}} \quad \left(= \frac{1}{\operatorname{rk} \mathcal{F}} \int_X c_1(\det \mathcal{F}) \wedge \omega^{n-1} \right)$$

A torsion-free coherent sheaf $\mathcal{F}$ (e.g., a vector bundle) is (semi)stable if

$$\mu(\mathcal{E}) \ (\leq) < \mu(\mathcal{F})$$

for all proper subsheaves $\mathcal{E}$ of $\mathcal{F}$.

Definition

A vector bundle E on X is curve semistable if for all morphisms $f: C \to X$ (where C is a smooth irreducible projective curve) the pullback bundle f^*E is semistable.

Theorem (Nakayama; B - Hernández Ruipérez)

Let E be a vector bundle on X. The following conditions are equivalent:

- *E is curve semistable;*
- *E is semistable and $\Delta(E) = 0$.*

$$\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 \in H^4(X, \mathbb{Q})$$

Simple fact: A bundle E with vanishing first Chern class is curve semistable if and only if it is nflat.

The S-fundamental group

- If $f: E \to F$ is a morphism of nflat bundles, $\ker(f)$ and $\mathrm{im}(f)$ are both nflat bundles, and $\mathrm{rk}(f)$ is constant
- the tensor product of nflat bundles is nflat

Let $\mathbf{NF}(X)$ be the category of nflat bundles on X, fix $x \in X$ and define a functor

$$\varpi_x : \mathbf{NF}(X) \to \mathbf{Vect}, \qquad E \mapsto E_x$$

Theorem (Langer)

$(\mathbf{NF}(X), \varpi_x)$ *is a neutral Tannakian category*

$\rightsquigarrow$ $\pi_1^S(X,x)$ the S-fundamental group scheme (Langer).

Over $\mathbb{C}$ (and modulo a technical condition[1]) $\pi_1^S(X,x)$ coincides with Simpson's universal complex fundamental group, which carries information on all finite-dimensional representations of the topological fundamental group.

[1] the cotangent sheaf contains no subsheaves of nonnegative slope

Grasmmann bundle

E vector bundle on X, $\mathrm{Gr}_k(E) \xrightarrow{\pi} X$ its k-th Grassmann bundle (bundle of k-planes in E)

$Y \xrightarrow{f} X, \qquad f^*E \to F \to 0$

$F = g^* \underset{\uparrow}{Q_k}$

rank k universal quotient bundle

$$\begin{tikzcd} & \mathrm{Gr}_k(E) \arrow[d, "\pi"] \\ Y \arrow[ur, dashed, "g"] \arrow[r, "f"'] & X \end{tikzcd}$$

$$0 \to S_k \to \pi^*E \to Q_k \to 0$$

If E is nef, all universal quotients bundles are nef (and viceversa, as $\mathrm{Gr}_1(E) = \mathbb{P}E$ and $Q_1 = \mathcal{O}_{\mathbb{P}E}(1)$) $\rightsquigarrow$ use this to get a definition of "numerical effectiveness" for Higgs bundles

Higgs Grasssmannians

$$\begin{tikzcd}
0 \arrow[r] & S_k \arrow[r] \arrow[d, dashed] & \pi^* E \arrow[r] \arrow[d, "\pi^*\phi"] & Q_k \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & S_k \otimes \Omega^1_{\mathrm{Gr}_k(E)} \arrow[r] & \pi^* E \otimes \Omega^1_{\mathrm{Gr}_k(E)} \arrow[r] & Q_k \otimes \Omega^1_{\mathrm{Gr}_k(E)} \arrow[r] & 0
\end{tikzcd}$$

(morphism of exact sequences of vector bundles on $\mathrm{Gr}_k(E)$)

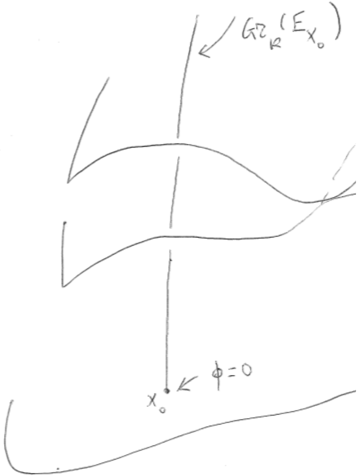
Definition

The k-th Higgs Grassmannian of the Higgs bundle $\mathfrak{E} = (E, \phi)$ is the closed subscheme

$$\mathfrak{Gr}_k(\mathfrak{E}) \subset \mathrm{Gr}_k(E)$$

where the the composition of the above blue arrows vanishes

$\mathfrak{Gr}_k(\mathfrak{E})$ is a very nasty scheme (it can be singular, nonreduced, non-equidimensional, ...)



Definition

If $\mathfrak{E} = (E, \phi)$ *is a Higgs bundle, a quotient* Q *of* E *is a* Higgs quotient *if the corresponding kernel* F *is* ϕ*-invariant,* $\phi(F) \subset F \otimes \Omega^1_X$.

Fact: Q is a Higgs quotient if and only if the corresponding section σ_Q of $\mathrm{Gr}_k(E)$ takes values in $\mathfrak{Gr}_k(\mathfrak{E}) \subset \mathrm{Gr}_k(E)$.

The universal quotient Q_k restricted to $\mathfrak{Gr}_k(\mathfrak{E})$ has a natural Higgs field Φ_k, so that we have a Higgs bundle $\mathfrak{Q}_{k,\mathfrak{E}} = (Q_{k|\mathfrak{Gr}_k(\mathfrak{E})}, \Phi_k)$ on $\mathfrak{Gr}_k(\mathfrak{E})$, and

$$\mathfrak{Q} = (Q, \phi_Q) = \sigma_Q^* \mathfrak{Q}_{k,\mathfrak{E}}.$$

Numerically flat Higgs bundles

(B., Graña Otero, Hernández Ruipérez)

Let $\mathfrak{E} = (E, \phi)$ be a Higgs bundle.

Definition

If $\operatorname{rk} E = 1$, $\mathfrak{E}$ *is H-nef if* E *is nef in the usual sense.*

If $\operatorname{rk} E > 1$, $\mathfrak{E}$ *is H-nef if*

- $\det E$ *is nef in the usual sense*
- *Every universal quotient Higgs bundle* $\mathfrak{Q}_k$ *is H-nef.*

$\mathfrak{E}$ *is H-nflat if both* $\mathfrak{E}$ *and* $\mathfrak{E}^*$ *are H-nef.*

A notion of semistability of Higgs bundles is introduced as for usual bundles, but checking the inequality between the slopes only for Higgs-invariant subsheaves.

Definition

A Higgs bundle $\mathfrak{E}$ on X is curve semistable *if for all morphisms $f: C \to X$ the pullback Higgs bundle $f^*\mathfrak{E}$ is semistable.*

Theorem (B - Hernández Ruipérez - Graña Otero, 2004, 2006)

If $\mathfrak{E}$ is semistable (with respect to some polarization) and $\Delta(E) = 0$, then it is curve semistable.

The opposite implication is a conjecture (B – Graña Otero, 2010).

Proposition

The previous conjecture is equivalent to the following statement: all Chern classes of an H-nflat Higgs bundle vanish.

Work done with A. Lo Giudice and V. Lanza shows that the conjecture holds for some classes of varieties, e.g.

- varieties whose tangent bundle is nef
- K3 surfaces (partial results) $\rightsquigarrow$ see Valeriano's talk

and for some close relatives.

The challenge is to prove the conjecture for varieties of general type

The fundamental Higgs scheme

Some properties of H-nflat Higgs bundles:

- a morphism of H-nflat Higgs bundles has constant rank;
- the kernel and cokernel of a morphism of H-nflat Higgs bundles are H-nflat Higgs bundles;
- the tensor product of two H-nflat Higgs bundles is H-nflat.

Theorem (Biswas - B - Gurjar, 2016)

The category $\mathbf{HNF}(X)$ *of H-nflat Higgs bundles on* X *is a neutral Tannakian category*

We denote by $\pi_1^H(X,x)$ the group scheme which represents it (the Higgs fundamental group scheme of X).

Since $\mathbf{NF}(X) \subset \mathbf{HNF}(X)$, there is a surjection

$$\pi_1^H(X,x) \twoheadrightarrow \pi_1^S(X,x)$$

HFGS vs. BGO's conjecture

- if $\pi_1^H(X,x) = \{e\}$ then $\mathbf{HNF}(X) = \mathbf{Vect}$, so that all H-nflat Higgs bundles are trivial, and the conjecture holds for X.

  Example: $\mathbb{P}^n$. Indeed the conjecture holds for $\mathbb{P}^n$, so that H-nflat Higgs bundles (E,ϕ) have vanishing Chern classes. Moreover E is semistable as a usual bundle, and then $\phi = 0$. As a result, E is trivial.
- if $\pi_1^H(X,x) = \pi_1^S(X,x)$ then $\mathbf{HNF}(X) = \mathbf{NF}(X)$, and all H-nflat Higgs bundles are nflat, so that the conjecture holds for X

If X and Y are varieties, there is a functor

$$\mathbf{HNF}(X) \times \mathbf{HNF}(Y) \to \mathbf{HNF}(X \times Y)$$

$$((E,\theta),(F,\phi)) \mapsto (E \boxtimes F, \theta \otimes \mathrm{id} + \mathrm{id} \otimes \phi)$$

so there is a morphism

$$\pi_1^H(X \times Y,(x,y)) \to \pi_1^H(X,x) \times \pi_1^H(Y,y) \qquad (*)$$

We do not know if this is an isomorphism. This is related to the conjecture: indeed, if (*) is an isomorphism (actually injective, at least for products of curves), then the conjecture holds for products of curves.