Flat connections on 2-manifolds

Introduction

Outline:

1. (a) The Jacobian (the simplest prototype for the class of objects treated throughout the paper) corresponding to the group $U(1)$).
   (b) SU(n) character varieties
   (c) Vector bundles of nonzero degree
   (d) Applications
2. (a) General properties
   (b) spaces of connections.
   (c) cohomology: $U(1)$ case
   (d) General case.
3. Recent work on the topology of these spaces

I Background Material

Let Σ be a compact two-dimensional orientable manifold. Unless otherwise specified, the dimension refers to the dimension as a *real* manifold. The space Σ can be described in different ways depending on how much structure we choose to specify.

(1) Topological description:

One may form a class of topological spaces homeomorphic to Σ by gluing together (in pairs) the edges of a polygon with $4g$ sides. These spaces are classified by their fundamental groups (in other words, by the genus g, for which the Euler characteristic of the space is $2-2g$: a space with genus g is a g-holed torus):

$$\pi = \pi_1(\Sigma^g) = \langle a_1, b_1, \ldots, a_g, b_g : \prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} = 1 \rangle.$$

The a_j, b_j provide a basis of $H_1(\Sigma)$, chosen so that their intersection numbers are

$$a_j \cap b_j = 1$$

and all other intersections are zero.

(2) Smooth description: The objects described in (1) may be endowed with structures of smooth orientable manifolds of dimension 2, and all smooth structures on a compact orientable 2-manifold of genus g are equivalent up to diffeomorphism.

(3) Holomorphic description: The objects in (1) and (2) may be endowed with additional structure, since they may be given a structure of complex manifold or Riemann surface (Riemann surfaces are complex manifolds of complex dimension 1). The collection of possible complex structures on an orientable 2-manifold of genus g is a complex variety called the *moduli space of Riemann surfaces of genus* g. For genus $g = 1$, the collection of complex structures on the 2-torus is the quotient of the upper half plane $\mathbb{H}$ by the natural action of $SL(2, \mathbb{Z})$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : z \mapsto \frac{az + b}{cz + d}.$$

The complex structure on any orientable 2-manifold of genus $g = 1$ is specified by the quotient of $\mathbb{C}$ by the lattice generated by 1 and τ where $\tau \in \mathbf{H}$. The collection of complex structures on an orientable 2-manifold of genus $g \geq 2$ is parametrized by the points in the quotient of $\mathbb{C}^{3g-3}$ by the action of a discrete group.

The Jacobian

One may associate certain spaces with Σ which admit different descriptions depending on the amount of structure with which we have equipped our 2-manifold. In algebraic geometry the moduli problem refers to the problem of describing the parameters on which a collection of algebraic varieties depend.

A prototype is the *Jacobian*, which can be described in several different ways.

(1) Topological description: If we view Σ as a topological space and retain only the structure of its fundamental group π, we may define

$$\mathrm{Jac}(\Sigma) = \mathrm{Hom}(\pi, U(1)) = U(1)^{2g}$$

(2) Smooth description: If we view Σ as a smooth orientable 2-manifold, the Jacobian has a gauge theory description

$$\mathrm{Jac}(\Sigma) = \text{ flat } U(1) \text{ connections } / \text{gauge group } .$$

The space of $U(1)$ connections is the space $\Omega^1(\Sigma)$ of 1-forms on Σ. In terms of local coordinates y_1, y_2 on a neighbourhood U in Σ, $\mathcal{A}$ is identified with two copies of $C^\infty(U)$.

We can impose the condition that a connection A be flat: this leads to the space $\mathcal{A}_{\text{flat}}$. The *gauge group* is $\mathcal{G} = C^\infty(\Sigma, U(1))$; its Lie algebra is $\mathrm{Lie}(\mathcal{G}) = C^\infty(\Sigma)$.

(3) Holomorphic description: If we endow Σ with a complex structure, $\mathrm{Jac}(\Sigma)$ is identified with an algebraic variety which classifies *holomorphic line bundles* over Σ: this is how the Jacobian arises naturally in algebraic geometry. Strictly speaking the term "Jacobian" refers only to the algebraic-geometric object, which is a complex torus.

Representations into a nonabelian Lie group

When we replace $U(1)$ by a compact *nonabelian* group G (e.g. $G = SU(2)$, or more generally $G = SU(n)$), some complications arise.

(1) Topological description:

The natural generalization of $\mathrm{Jac}(\Sigma)$ is

$$\mathcal{M}(\Sigma) = \mathrm{Hom}(\pi, G)/G$$

where G acts on $\mathrm{Hom}(\pi, G)$ by conjugation.

(2) Smooth description: $\mathcal{M}(\Sigma)$ has a natural gauge theory description which generalizes the description of the Jacobian:

$$\mathcal{M}(\Sigma) \cong \text{flat } G \text{ connections on } \Sigma/\mathcal{G}$$

where $\mathcal{G} = C^\infty(\Sigma, G)$ is the gauge group.

(3) Holomorphic description: $\mathcal{M}(\Sigma)$ also has a description in algebraic geometry: it is the moduli space of holomorphic $G^{\mathbb{C}}$ bundles over Σ, with an appropriate stability condition from geometric invariant theory. The identification between the space of representations and that of holomorphic vector bundles was established by Narasimhan and Seshadri (1965).

Example: For $G = U(n)$, $\mathcal{M}(\Sigma)$ is identified with the moduli space of (semistable) holomorphic vector bundles of rank n and degree 0 over Σ. When $G = SU(n)$ we obtain the moduli space of (semistable) holomorphic vector bundles of rank n with trivial determinant line bundle over Σ.

Vector bundles of nonzero degree

More generally, we can consider the spaces $M(n, d)$ of (semistable) holomorphic vector bundles of rank n and (possibly nonzero) degree d with fixed determinant line bundle L over Σ. Replace π by

$$\pi' = \pi_1(\Sigma \setminus D) = \{x_1, \ldots, x_{2g}\}$$

(the free group on $2g$ generators), where D is a small disc in Σ. The group π' is the fundamental group of a compact orientable 2-manifold with one boundary component, which may be obtained by removing D from Σ.

For $G = SU(n)$, choose $c = e^{2\pi i d/n}\mathbb{I}$ which generates the centre $Z(G)$ (in other words n and d are coprime). For example, when $G = SU(2)$ we choose $c = -\mathbb{I}$.

(1) Topological description:

$$M(n,d) = \left\{ \rho \in \mathrm{Hom}(\pi', G) : \rho(\prod_{j=1}^{g} x_{2j-1} x_{2j} x_{2j-1}^{-1} x_{2j}^{-1}) = c \right\} / G$$

where G acts by conjugation.

(2) Smooth description: $M(n, d)$ has a description as the space of gauge equivalence classes of flat G connections on $\Sigma - D$, whose holonomy around the boundary of D is conjugate to $e^{2\pi i d/n}\mathbb{I}$.

(3) Holomorphic description: $M(n,d)$ is the space of (semistable) holomorphic vector bundles of rank n, degree d and fixed determinant over Σ . Provided n and d are coprime, the space $M(n,d)$ is a smooth manifold with a symplectic form.

The space $M(n,d)$ is also a complex manifold, and the complex structure is compatible with the symplectic form. In other words, $M(n,d)$ is a Kähler manifold.

Importance of these spaces

Spaces of flat connections on oriented 2-manifolds arise in a number of different contexts:

1. **Gauge theory**: The properties of these spaces are a prototype for properties of topological spaces arising in gauge theory related to manifolds of dimension higher than 2 (for example Donaldson or Seiberg-Witten invariants in dimension 4 and Floer homology in dimension 3.
2. **Topology:** These spaces provide a natural setting for various questions involving topology of manifolds of dimension 2 and 3. For example, the Casson invariant is an invariant of 3-manifolds; it arises naturally as the intersection number of two Lagrangian submanifolds in a space of flat connections.
3. **Mathematical physics:** These spaces arise from the study of the *Yang-Mills equations* on a manifold of dimension 2.
4. **Algebraic geometry:** Moduli spaces of holomorphic vector bundles on Riemann surfaces have surprising properties in the context of algebraic geometry. For example, the *Verlinde formula* is a formula for the dimension of the space of holomorphic sections of a line bundle $\mathcal{L}$ over $M(n, d)$. Drezet-Narasimhan: all holomorphic line bundles $\mathcal{L}$ arise as powers $\mathcal{L} = \mathcal{L}_0^k$ of

a generating line bundle $\mathcal{L}_0$ over $M(n,d)$.

The Verlinde formula is remarkable since it is often difficult to explicitly determine the number of holomorphic sections of a bundle over a complex manifold.

5. **Symplectic geometry:** Spaces of flat connections on oriented 2-manifolds are *symplectic manifolds* and may be studied from that point of view.

6. **Relation to symplectic and geometric invariant theory quotients:**

• Atiyah and Bott (1982) exhibited the spaces $M(n,d)$ as *symplectic quotients* via an infinite dimensional construction; the space of all connections $\mathcal{A}$ on Σ is acted on by the gauge group $\mathcal{G}$ with moment map the curvature

$$\mu : A \mapsto F_A.$$

• Thus the symplectic quotient (in other words $\mu^{-1}(0)/\mathcal{G}$) is the space of flat connections up to equivalence under the action of the gauge group.

• These spaces are interesting examples of quotient constructions in symplectic geometry and geometric invariant theory. It has been established (see for instance Mumford-Fogarty-Kirwan Chapter 8) that the symplectic quotient of a Kähler manifold by a (compact) group G is equivalent to the geometric invariant theory quotient by the complexification $G^{\mathbb{C}}$.

Spaces of flat connections on oriented 2-manifolds

General properties

1. Flat connections on Σ:

   If the bundle can be trivialized, it is equivalent to the product bundle $\Sigma \times G$.

   There are many different ways to specify the trivialization. After one fixed choice of trivialization has been made, the choice of another trivialization is equivalent to the choice of an element of the *gauge group*

   $$\mathcal{G} = C^\infty(\Sigma, G),$$

   the group of (smooth) maps from Σ to G.

2. The Lie algebra of the gauge group is the space of smooth maps from Σ to $\mathrm{Lie}(G)$. The gauge group acts on the space of connections in the following way: if G is a matrix group, a choice of a trivialization of the principal bundle identifies the space $\mathcal{A}$ of connections with $\Omega^1(\Sigma) \otimes \mathrm{Lie}(G)$, and an element Γ in the gauge group sends $A \in \mathcal{A}$ to $\Gamma^{-1} A \Gamma + \Gamma^{-1} d\Gamma$.

FACT:

$$\frac{\{\text{ flat connections}\}}{\text{gauge group}} \cong \frac{\mathrm{Hom}(\pi, G)}{G}$$

where G acts on $\mathrm{Hom}(\pi, G)$ by conjugation .

The space described above may be called the space $\mathcal{M}$ of flat connections modulo gauge transformations, equivalently the space of representations of the fundamental group of Σ modulo conjugation.

Special cases:

Example 1: The torus $S^1 \times S^1$

The fundamental group of the torus is generated by two loops a and b and they commute with each other. Thus the fundamental group is commutative.

More generally, an oriented 2-manifold of genus g (g-holed torus) is formed by taking a polygon with $4g$ sides and gluing the sides together in pairs The sides of the polygon become the generators of the group.

Now, however, the group is not commutative: from the information that the loop around the outside of the polygon can be shrunk to a point we learn only that the generators satisfy the relation

$$a_1 b_1 (a_1)^{-1} (b_1)^{-1} \dots a_g b_g (a_g)^{-1} (b_g)^{-1} = 1.$$

In order to specify a representation ρ of π into a compact Lie group G we must specify the elements A_i, B_i in G to which ρ sends each loop a_i, b_i. In order that it should be a representation we insist that the relation is preserved:

$$A_1 B_1 (A_1)^{-1} (B_1)^{-1} \dots A_g B_g (A_g)^{-1} (B_g)^{-1} = 1.$$

We must also take the quotient by the action of G by conjugation on the space of representations:

$$h \in G : A_i \mapsto h^{-1} A_i h; \; B_i \mapsto h^{-1} B_i h.$$

Example 2: $\Sigma = S^1 \times S^1$, $G = U(n)$ (a nonabelian group).

The fundamental group is abelian, so we obtain

$$\mathcal{M} = (T \times T)/W \tag{1}$$

where T is the maximal torus of G and W is the Weyl group.

Example 3: $G = U(1)$, the circle group. Note that this group is commutative, so the conjugation action is the identity map

$$A_i \mapsto A_i, B_i \mapsto B_i$$

for any $g \in G$. Also, any elements $A_i, B_,i$ of G automatically satisfy the above relation because $A_i B_i (A_i)^{-1} (B_i)^{-1} = 1$ for any A_i and B_i.

So for this group the space $\mathcal{M}$ is simply $U(1)^{2g}$.

Example 4: The general case

Let $G = U(n)$, and suppose Σ is an orientable 2-manifold with genus $g > 1$. In the case when G is a nonabelian group (such as $U(n)$ when $n > 1$) the space is more complicated than when G is abelian. Two differences are apparent:

a. The relation between the images of the generators (imposed by the fact that the loop around the boundary of the polyhedron can be shrunk to a point) is no longer automatically satisfied.

b. The action of the group on the space of representations by conjugation is now nontrivial. In this case the space $\mathcal{M}$ is usually not smooth.

A smooth analogue is formed as follows:

• Remove a small disc in Σ • Require the representation to send the loop around the boundary of the disc to $e^{2\pi i d/n}$ (in terms of vector bundles, n is the rank and d is the degree). We assume these are relatively prime.

This space (denoted $M(n,d)$) is in fact smooth, and shares many properties with the more natural space $\mathcal{M}$.

Connections

• The space $\mathcal{A}$ of all connections is simply the vector space of 1-forms tensored with $\mathbf{g}$. To determine the tangent space to the space of *flat* connections, we use the fact that the curvature is the quantity

$$F_A = dA + \frac{1}{2}[A, A].$$

• Infinitesimally, if $F_A = 0$ then the condition that $F_{A+a} = 0$ translates to

$$da + [A, a] = 0.$$

We write this as

$$d_A a = 0.$$

It turns out that one can generalize this definition (in a way that can easily be described) to give an operator d_A which maps $\mathbf{g}$-valued differential forms of degree p to $\mathbf{g}$-valued differential forms of degree $p + 1$ and satisfies

$$d_A \circ d_A = 0.$$

• At the infinitesimal level, the image of the $\mathbf{g}$-valued 0-forms under d_A is the tangent space to the orbits of the group of gauge transformations. Thus the tangent space to the space $\mathcal{M}$ is the space

$$T_A\mathcal{M} = H^1(\Sigma, d_A) = \frac{\{a \in \Omega^1(\Sigma) \otimes \mathbf{g} \ \mid \ d_A a = 0\}}{\{d_A\phi \mid \phi \in \Omega^0(\Sigma) \otimes \mathbf{g}\}}.$$

• The space $\mathcal{M}$ has a *symplectic form*, a closed 2-form which induces a nondegenerate skew-symmetric pairing on each tangent space. At the level of the vector space $\mathcal{A}$ of all connections, this just comes from the wedge product on differential forms, combined with an Ad-invariant inner product $\langle , \rangle$ on $\mathbf{g}$; choosing a basis $\{e_\alpha\}$ for $\mathbf{g}$ we have

$$\omega(A, B) = \sum_{\alpha,\beta} \langle e_\alpha, e_\beta \rangle \int_\Sigma A^\alpha \wedge B^\beta.$$

Here we have written $A = \sum_\alpha A^\alpha e_\alpha$ for 1-forms A^α (and similarly for B). One can see (using Stokes' theorem) that ω descends to a skew-symmetric pairing on $H^1(\Sigma, d_A)$ (because $\omega(d_A\phi, b) = 0$ for all $b \in \Omega^1(\Sigma, \mathbf{g})$ and any $\phi \in \Omega^0(\Sigma, \mathbf{g})$.) In fact this pairing is nondegenerate.

• If G is abelian, this pairing is just the cup product on cohomology, because in de Rham cohomology the cup product is represented by the wedge product of differential forms. This specifies a symplectic structure on $\mathcal{M}$.

This symplectic structure was constructed by Goldman 1984, Karshon 1992 and

Weinstein 1995.

Cohomology of $U(1)$ spaces

$G = U(1)$:

1. A connection A is simply a 1-form $\sum_{i=1}^{2} A_i dx^i$ on Σ.
2. The connection A is flat if and only if $dA = 0$ in terms of the exterior differential d (which sends p-forms to $(p+1)$-forms).
3. The connection resulting from the action of an infinitesimal gauge transformation ϕ (where ϕ is a $\mathbf{R}$-valued function on Σ) is the 1-form $d\phi$.
4. flat connections /gauge $\cong H^1(\Sigma, \mathbf{R})/\mathbb{Z}^{\mathbf{2g}}$.

   We have

$$\begin{aligned} &\mathcal{A}_{\text{flat}}/\exp \mathrm{Lie}(\mathcal{G}) \\ &= \mathbf{R}^{\mathbf{2g}} \end{aligned}$$

   We must divide by an additional $\mathbb{Z}^{2g}$.

$$\mathcal{A}_F/\mathcal{G} \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g} \cong (U(1))^{2g}.$$

5. Thus the cohomology of the space $U(1)^{2g}$ has $2g$ generators $d\theta_i$, $i = 1, \ldots, 2g$ and the only relations are that

$$d\theta_i \wedge d\theta_j = -d\theta_j \wedge d\theta_i, \quad i, j = 1, \ldots, 2g$$

II. Cohomology: the general case

• How to find analogous generators and relations for the case of $\mathcal{M}$ when G is a nonabelian group such as $U(n)$ or $SU(n)$?

Here the generators of the cohomology ring are obtained as follows.

1. There is a vector bundle $\mathcal{U}$ (the "universal bundle") over $\mathcal{M} \times \Sigma$. The bundle $\mathcal{U}$ has a structure of holomorphic bundle over $\mathcal{M} \times \Sigma$, such that its restriction to $\{x\} \times \Sigma$ for any point $x \in \mathcal{M}$ is the holomorphic vector bundle over Σ parametrized by the point x.

2. Take a connection A on $\mathcal{U}$ and decompose polynomials in its curvature F_A (for example $\mathrm{Trace}(F_A^n)$) into the product of closed forms on Σ and closed forms on $\mathcal{M}$.

3. integrate these forms over cycles in Σ (a point or 0-cycle, the 1-cycles a_i and b_i, or the 2-cycle given by the entire 2-manifold Σ) to produce closed forms on $\mathcal{M}$, which represent the generators of the cohomology ring of $\mathcal{M}$.

   • These classes generate the cohomology of $\mathcal{M}$ under addition and multiplication. Identifying the relations between these generators is much more difficult than in the $U(1)$ case.

One important cohomology class is the cohomology class of the Kähler form on $\mathcal{M}$. The Kähler class is the cohomology class $[\Sigma]$ obtained by taking the slant product of $c_2(\mathcal{U})$ with the fundamental class of Σ. Another important family of classes are those obtained by evaluating the classes on $\mathcal{M} \times \Sigma$ at a point in Σ. For the space $M(2,1)$ the class obtained by evaluating $c_2(\mathcal{U})$ at a point in Σ is often denoted $a \in H^2(M(2,1))$. The class in (22) is often denoted $f \in H^4(M(2,1))$, and it is the class arising from $c_2(\mathcal{U})$ evaluated on the fundamental class of Σ. This class is frequently chosen as the normalization of the cohomology class of the symplectic form on $M(2,1)$. Newstead (1972) describes the generators of this cohomology ring. The relations were established by Thaddeus (1991).

III. Witten's formulas

Witten (1989, 1992) obtained formulas for intersection numbers in the cohomology of these spaces $M(n,d)$. In particular, he obtained formulas for their symplectic volumes.

For $SU(2)$ these formulas are as follows.

Example: $n=2, d=1$ (Donaldson, Thaddeus 1991) • In this case the cohomology is generated by the generators $a \in H^4(M(2,1))$, $f \in H^2(M(2,1))$ and $b_j \in H^3(M(2,1)), j=1,\ldots,2g$. The structure of the cohomology ring is then determined by the relations between these generators. Since the cohomology of a compact manifold satisfies Poincaré duality, these relations are determined by the intersection numbers of all monomials in the generators.

• Donaldson and Thaddeus showed one may eliminate the odd degree generators b_j (see), so the structure of the cohomology ring can be reduced to knowing the intersection numbers of all powers of the two even degree generators a and f described above.

$$\int_{M(2,1)} a^j \exp f$$

$$= \frac{(-1)^j}{2^{g-2}\pi^{2(g-1-j)}} \sum_{n\rangle 0} \frac{(-1)^{n+1}}{n^{2g-2-2j}}$$

$$= \frac{(-1)^j}{2^{g-2}\pi^{2(g-1-j)}} (1 - 2^{2g-3-2j})\zeta(2g-2-2j)$$

Here we have used the notation

$$\exp f = \sum_{m \geq 0} \frac{f^m}{m!}$$

and we use the fact that $\int_{M(2,1)} \alpha = 0$ (where the integral denotes evaluation on the fundamental class of $M(2,1)$) unless the degree of α equals the dimension of $M(2,1)$.

• We note that the formulas for intersection numbers can be written in terms of a sum over irreducible representations of G: this is the form in which these formulas appeared in Witten's work.

Example: The symplectic volume of the space $\mathcal{M}$ of gauge equivalence classes of flat G connections is given by the "Witten zeta function":

$$\int_{\mathcal{M}} \exp(f) \sim \sum_R \frac{1}{(\dim R)^{2g-2}}$$

where we sum over irreducible representations R of G. In the preceding formula and the next two formulas, the symbol $\sim$ means that the left hand side is proportional to the right hand side by a known proportionality constant. For the details, see Witten (1992). In the special case of $SU(2)$ we have

$$\int_{\mathcal{M}} \exp(f) \sim \sum_n \frac{1}{n^{2g-2}}$$

and

$$\int_{M(2,1)} \exp(f) \sim \sum_n \frac{(-1)^{n+1}}{n^{2g-2}}$$

where we sum over the irreducible representations of $SU(2)$, which are parametrized by their dimensions n. In this case the Witten zeta function reduces to the Riemann zeta function.

Witten (1989) expresses the volume in terms of Reidemeister-Ray-Singer torsion, and gives a mathematically rigorous argument calculating it.

He also gives several physical arguments, including one which relies on the asymptotics of the Verlinde formula and another that refers to earlier work of Migdal.

IV.Mathematical proof of Witten's formulas

The space $\mathcal{M}$ is a symplectic quotient

$$\mu^{-1}(0)/\mathcal{G},$$

where μ is the moment map (a collection of Hamiltonian functions whose Hamiltonian flows generate the action of a group $\mathcal{G}$ on a symplectic manifold M):

1. The space $\mathcal{M}$ may be constructed as an infinite-dimensional symplectic quotient of the space of all connections $\mathcal{A}$ by the gauge group $\mathcal{G}$: the moment map $\mu : \mathcal{A} \to \mathrm{Lie}(\mathcal{G})^*$ of a connection A is its curvature $\mu(A) = F_A \in \Omega^2(\Sigma, \mathbf{g})$ $= \Omega^0(\Sigma, \mathbf{g})^*$ (and $\Omega^0(\Sigma, \mathbf{g})$ is the Lie algebra of the gauge group). Hence $\mu^{-1}(0)/\mathcal{G}$ is the space $\mathcal{M}$.

2. The space $\mathcal{M}$ may also be constructed as a finite-dimensional symplectic quotient of a (finite dimensional) space of flat connections on a punctured Riemann surface, by the action of the finite-dimensional group G.

   • This may involve an extended moduli space (LJ, 1994) or the quotient of a space with a group-valued moment map (Alekseev-Malkin-Meinrenken, 1998).

• We use formulas (J-Kirwan 1998) for intersection numbers in a symplectic quotient, in terms of the restriction to the fixed points of the action of a maximal commutative subgroup of G (for $G = U(n)$ this subgroup is the diagonal matrices $U(1)^n$). The answer is given in terms of:

1. the action of the maximal torus T on the normal bundle to the fixed point set to the T action;
2. the values of the moment map on the fixed point set;
3. the restriction of the cohomology classes to the fixed point set.

Using these methods we recover Witten's formulas.

V. Hamiltonian flows

Let $S_1, \ldots, S_{3g-3}$ be a collection of simple closed curves in a 2-manifold Σ of genus g.

Each curve induces a Hamiltonian flow on the moduli space.

For any two disjoint curves, the corresponding flows commute.

(Goldman 1986)

If the Hamiltonian $\mathrm{Trace}(\theta)$ is replaced by $\cos^{-1}(\mathrm{Trace}\theta/2)$ then the flows become periodic with constant period (they become the Hamiltonian flows for a Hamiltonian S^1 action).

VI. Geometric quantization of the SU(2) moduli space

(LJ and J. Weitsman, CMP 1993)

1. Geometric quantization replaces a symplectic manifold (the classical phase space) by a vector space with inner product (the physical Hilbert space).
2. If the symplectic manifold has dimension n, the quantization $\mathcal{H}$ should consist of 'functions of half the variables'

   Prototype: $M = \mathbf{R^2}$ with coordinates q (position) and p (momentum).
3. One way to do this is to let $\mathcal{H}$ be holomorphic functions in $p + iq$ (complex polarization).
4. Alternatively (Sniatycki 1980) we could use a real polarization (a map π to a manifold of half the dimension, with fibres Lagrangian submanifolds) and define $\mathcal{H}$ to be functions covariant constant along the fibres of the polarization (in the $\mathbf{R^2}$ example, this is like functions of p or q)
5. Geometric quantization with a real polarization gives a basis consisting of sections of the prequantum line bundle covariant constant along the fibres.

The choice of $3g - 3$ disjoint circles in a 2-manifold (in other words a pants decomposition) specifies a real polarization on $\mathcal{M}$.

The map sending a flat connection A to the angle θ_j for which the holonomy of A around C_j is conjugate to

$$\begin{bmatrix} e^{i\theta_j} & 0 \\ 0 & e^{-i\theta_j} \end{bmatrix}$$

is the moment map for a Hamiltonian circle action on an open dense subset of $\mathcal{A}$.

VII. Goldman flows

Goldman (1986) studied the Hamiltonian flows of the functions $A \mapsto \mathrm{Trace}(\mathrm{Hol}_{C_j} A)$ and found that these functions Poisson commute provided the curves C_j are disjoint.

In terms of flat connections, the Hamiltonian flows are given as follows. Let A be a flat connection on Σ.

• We assume a curve $C \subset \Sigma$ is chosen.

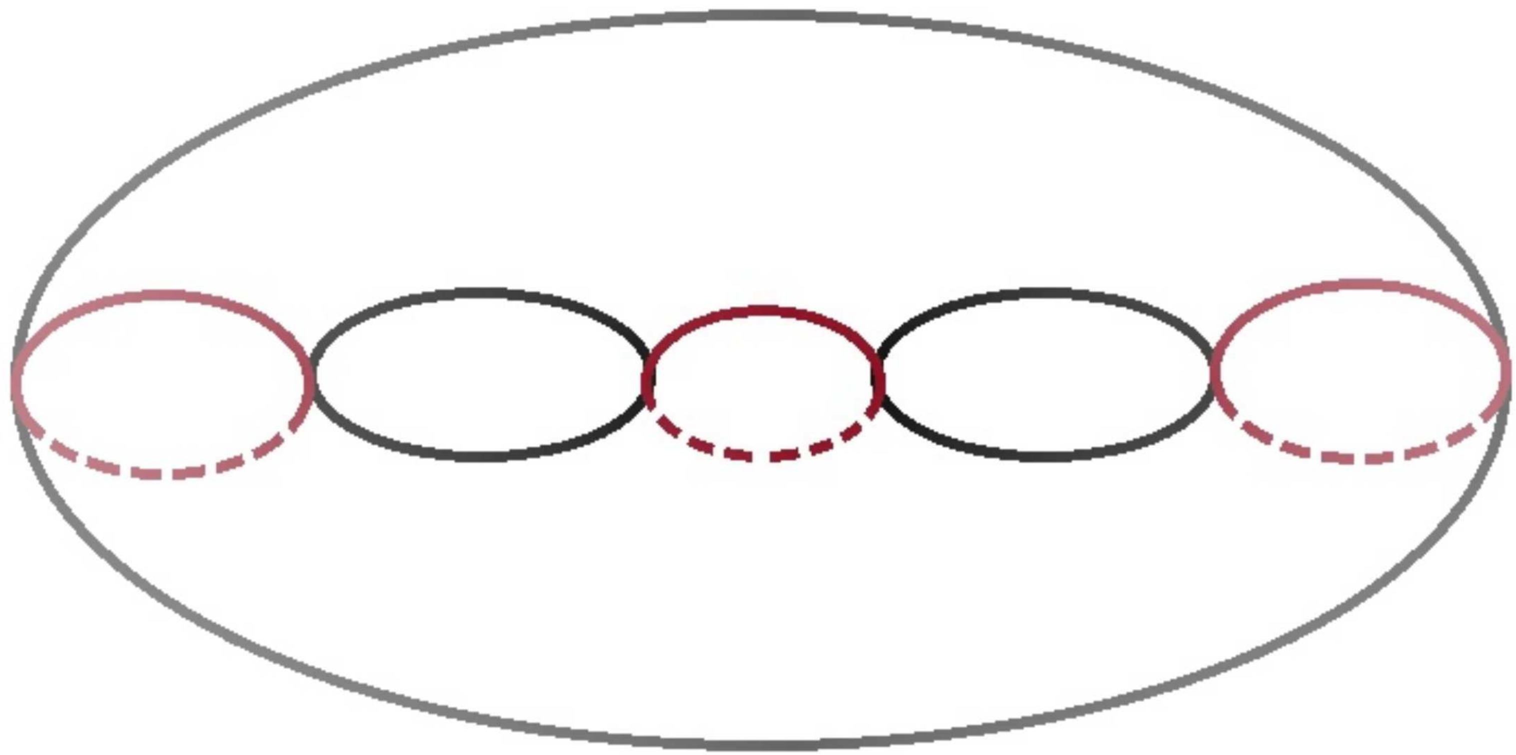
• Assume the holonomy of A around C is in a chosen maximal torus T (say the diagonal matrices in $SU(2)$).

• Define $\Sigma' = \Sigma \setminus C$. This has two boundary components C_+ and C_-.

• Define $e^{it}(A)$ to be the result of applying a gauge transformation over Σ' which is the identity on C_- and the diagonal matrix with eigenvalues $e^{\pm it}$ on C_+. The result is a flat connection on Σ (because its values on C_+ and C_- are equal), but it is not gauge equivalent to A (since the gauge transformation does not come from a gauge transformation on Σ). This defines an S^1 action on (an open dense set of) the space of gauge equivalence classes of flat connections on Σ.

• The circle action is not well defined when the stabilizer of the holonomy of A around C is larger than T, since there is no canonical way to choose a maximal torus containing this holonomy.

• For us, the flows are defined on the faces of the moment polytope as well as the interior of the moment polytope, but not along the edges or vertices. We equip the genus 2 surface with the following pants decomposition:



Pants decomposition of a genus 2 surface

The moment polytope is then a tetrahedron.

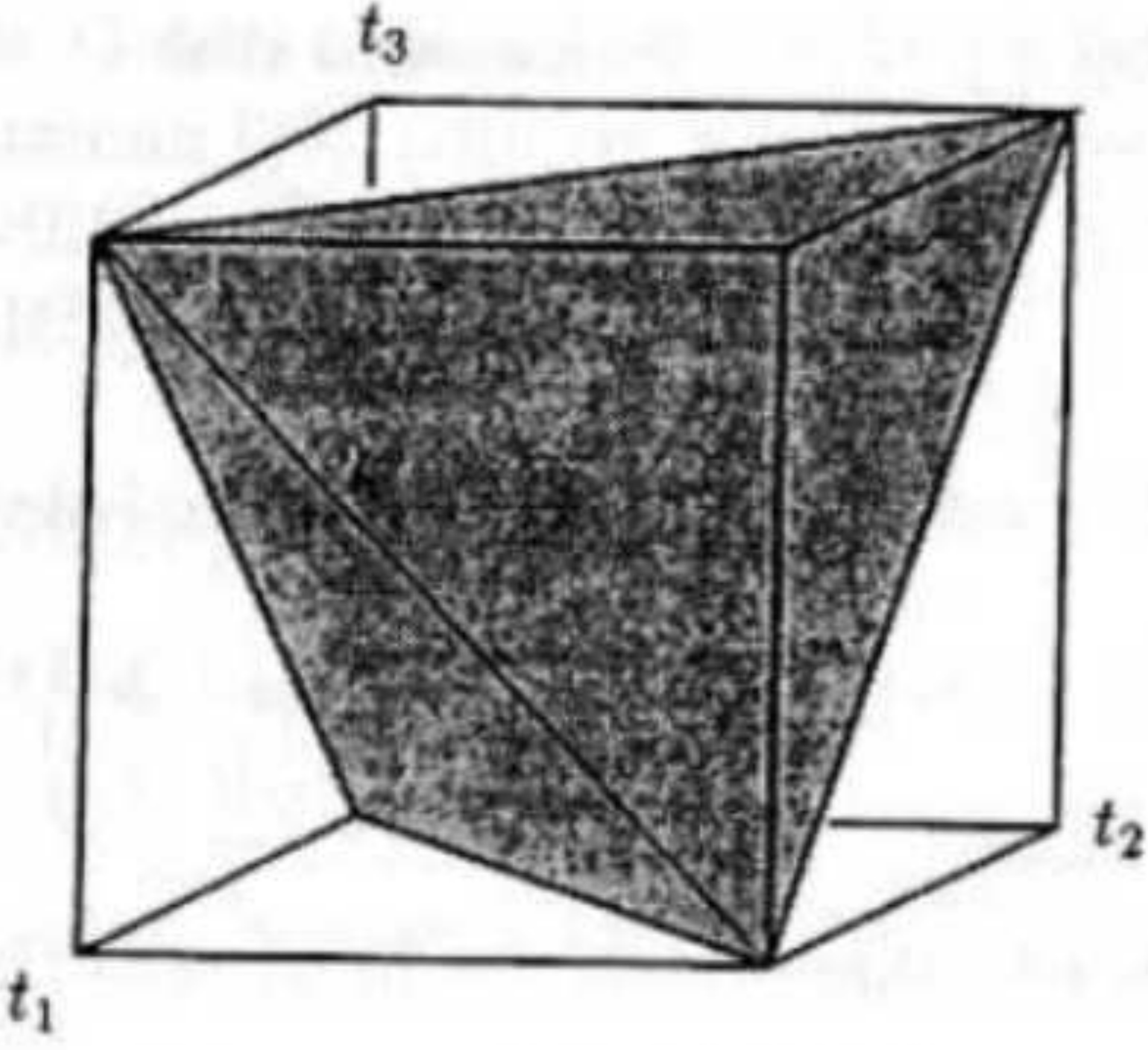
The holonomies around C_j are characterized by the inequalities

$$|\theta_i - \theta_j| \leq \theta_k \leq \theta_i + \theta_j$$

$$\theta_i + \theta_j + \theta_k \leq 2\pi$$

(where $0 \leq \theta_i \leq \pi$)

This set of inequalities specifies a tetrahedron.


t_3
t_2
t_1

The tetrahedron which is the image of the moment map

VIII. The Verlinde dimension formula

The Verlinde dimension is a formula for the space of holomorphic sections of $\mathcal{L}^k$ where $\mathcal{L}$ is the prequantum line bundle. Drezet-Narasimhan (1989) showed that all line bundles are powers of the prequantum line bundle.

The *Verlinde dimension formula* is the dimension of the space of holomorphic sections of $\mathcal{L}^r$ where $\mathcal{L}$ is the prequantum line bundle over $\mathcal{M}$. It is the number of labellings l_j of the curves C_j by integers in $[0, r]$ so that $(\frac{\pi l_i}{r}, \frac{\pi l_j}{r}, \frac{\pi l_k}{r})$ lies in the above tetrahedron whenever C_i, C_j, C_k are boundary circles of a pair of pants, and additionally

$$l_i + l_j + l_k \in 2\mathbb{Z}$$

LJ-J. Weitsman: The Verlinde formula is the dimension of the integer values of the moment map for the circle actions.

Recall from toric geometry: The dimension of the space of holomorphic sections of $\mathcal{L}^k$ is the number of integer points in the dilation of the moment polytope (Newton polytope) dilated by the factor k. (See for example W. Fulton, *Introduction to Toric Varieties*, 1993.

This is an example of independence of polarization (since the Verlinde formula was computed using a complex polarization, whereas we computed the dimension using a real polarization).

To get the correct answer we need to include points on the boundary of the moment polytope, where strictly speaking the Hamiltonian torus actions are not defined.

IX. Volumes of moduli spaces

(a) LJ and J. Weitsman, 'Inductive decompositions and vanishing theorems' Canad. J. Math. **52** (2000)

Let $\mathcal{M}_g(\Lambda)$ be the space of gauge equivalence classes of flat connections on surfaces with boundary where the holonomy is constrained to live in a particular conjugacy class specified by a parameter Λ in the fundamental Weyl chamber. We prove differential equations relating the volume in genus g to that in genus $g-1$:

$$(D_\Lambda)^2 S_{g-1}(\Lambda) = C S_{g-1}(\Lambda)$$

where $S_g(\Lambda)$ is the volume of $\mathcal{M}_g(\Lambda)$.

Here D_Λ is a differential operator of order n_+ (the number of positive roots).

Consequently we obtain vanishing theorems in the cohomology ring of these spaces (cf. the Newstead conjectures, which can be given a new proof by these methods).

Compare Adina Gamse and Jonathan Weitsman (2016)

(b) LJ and J. Weitsman, Toric structures on the moduli space of flat connections on a Riemann surface II: Inductive decomposition of the moduli space. *Math. Annalen* **307** (1997)

We use fibrations obtained from a pants decomposition of a surface to relate symplectic volumes of spaces of flat connections of genus g with symplectic volumes in genus $g-1$ (constructing the genus g surface by adding a handle to a genus $g-1$ surface). Some of the results of 'Inductive decompositions and vanishing theorems' are obtained this way.