Real loci in symplectic manifolds

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I. Background

Let M be a manifold equipped with a nondegenerate closed 2-form ω (the symplectic form).

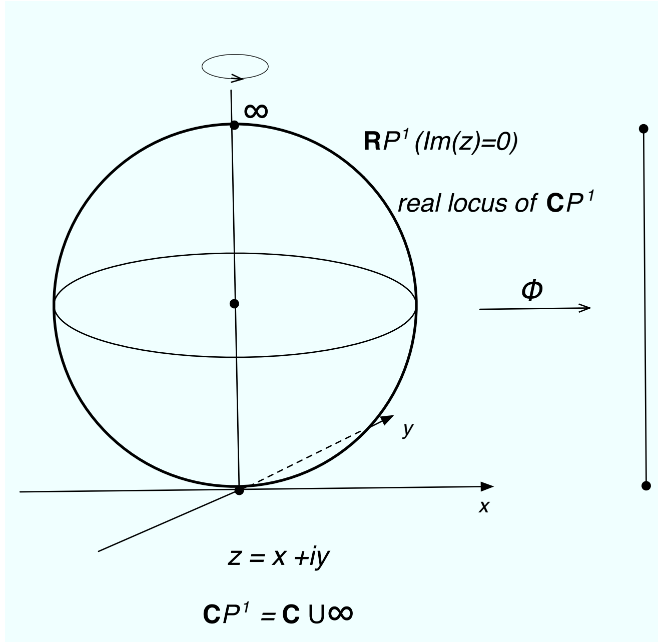
We assume that a Lie group G acts preserving ω, and that the action is obtained from the Hamiltonian flows of a collection of Hamiltonian functions (moment maps $\Phi : M \to \mathrm{Lie}(G)^*$).

Example

Examples: the orbits of the (co)adjoint action of G on its Lie algebra are symplectic manifolds, and the moment map is the inclusion into the Lie algebra.

Special case: $S^1 \subset SU(2)$ acts by rotation on S^2, which is the orbit of the adjoint action of $SU(2)$ on its Lie algebra.

Example 1: S^1 rotation action on $S^2 = \mathbb{C}P^1$



Two ingredients:

- rotation action $S^1 \curvearrowright \mathbb{C}P^1$ via $e^{i\theta}[z_1, z_2] = [z_1, e^{i\theta} z_2]$, moment map Φ (height function).

- automorphism τ of $\mathbb{C}P^1$ given by $\tau[z_1, z_2] = [\overline{z_1}, \overline{z_2}]$, fixed points $\mathbb{R}P^1$

Facts:

1. $\boxed{\Phi(\mathbb{C}P^1) = \Phi(\mathbb{R}P^1)}$

2. Cohomology rings:

- *ordinary cohomology*

$H^*(\mathbb{C}P^1; \mathbb{Z}_2) = \mathbb{Z}_2[c]/\langle c^2 \rangle, \deg c = 2$

$H^*(\mathbb{R}P^1; \mathbb{Z}_2) = \mathbb{Z}_2[w]/\langle w^2 \rangle, \deg w = 1$

$\Rightarrow \boxed{H^{2*}(\mathbb{C}P^1; \mathbb{Z}_2) \simeq H^*(\mathbb{R}P^1; \mathbb{Z}_2)}$

General Situation: A *conjugation space* [Hausmann, Holm, Puppe 2005] is a symplectic manifold (M, ω) with

- Hamiltonian torus action $T \curvearrowright M$, moment map $\Phi : M \to \mathfrak{t}^*$
- involution $\tau : M \to M$

$(\tau \circ \tau = \mathrm{id}_M)$

$\tau^* \omega = -\omega$ (τ is anti-symplectic)

τ compatible with T action [Duistermaat 1983]:

$\boxed{\tau(t.x) = t^{-1}.\tau(x), \ \forall t \in T, x \in M}$

Definition (O'Shea-Sjamaar): Let σ be an involution of T. The torus action is compatible with the involution if

$$\tau(ux) = \sigma(u)\tau(x)$$

for all $u \in T$ and $x \in M$.

(Duistermaat: σ is complex conjugation.)

Denote $M^\tau = \{x \in M \ : \ \tau(x) = x\}$ (*real locus*). The real locus is a Lagrangian submanifold of M (in other words ω restricts to 0 on it, and its dimension is half the dimension of M).

Examples (HHP 2005):

- coadjoint orbits (with the Chevalley involution)
- toric manifolds
- complex Grassmannians
- polygon spaces (for example Klyachko 1994, Kapovich-Millson 1996)
- $\mathbb{C}P^n$ with the standard action of $U(1)^n$ and the involution given by complex conjugation.

  The real locus is $\mathbb{R}P^n$.

Assume M is compact.

Theorem 1. (Duistermaat 1983)
$\Phi(M) = \Phi(M^\tau)$.

Theorem 2. (Hausmann-Holm-Puppe 2005) *Under some additional hypotheses (e.g. M^T discrete) we have ring isomorphisms*
$H^{2*}(M; \mathbb{Z}_2) \simeq H^*(M^\tau; \mathbb{Z}_2)$

Goldin-Holm (2004) studied the real locus of a symplectic reduced space under a torus action, and its image under the moment map.

II. Toric manifolds

Delzant's theorem: Toric manifolds: If M is a symplectic manifold of real dimension $2n$ admitting an effective Hamiltonian action of a torus of dimension n (a toric manifold), the image of the moment map is a convex polyhedron (the Delzant polytope). Any two such with the same moment polytope are diffeomorphic via a T-equivariant diffeomorphism that respects the moment maps.

If M is a toric manifold with a compatible antisymplectic involution, then Duistermaat's theorem asserts that the moment map maps the fixed point set of the involution onto the Delzant polytope. This is not necessarily a bijection though.

Example: projective space $\mathbb{C}P^n$ is equipped with the Hamiltonian torus action of $U(1)^n$ and the involution given by complex conjugation. The fixed point set is $\mathbb{R}P^n$. A simple examination of the fixed point set shows that the moment map is not a bijection between the fixed point set and the tetrahedron.

III. THE BASED LOOP GROUP

The based loop ΩG was studied by Pressley-Segal (1988).

Goal. Extend Theorems 1 and 2 to $M = \Omega G$ (based loops in the compact Lie group G)

Set-up:

- G is a simply connected simple compact Lie group, $T \subset G$ maximal torus
- $\Omega G = \{\gamma : S^1 \to G \ : \ \gamma(1) = e\}$
- T action on ΩG: $(t.\gamma)(z) = t\gamma(z)t^{-1}$
- S^1 action on ΩG: $(e^{i\theta}\gamma)(z) = \gamma(e^{i\theta}z)\left(\gamma(e^{i\theta})\right)^{-1}$ ("rotation" – note we must preserve the condition that $\gamma(1) = e$)

- $T \times S^1 \curvearrowright \Omega G$ is Hamiltonian, moment map $\Phi : \Omega(G) \to \mathrm{Lie}(T) \oplus i\mathbb{R}$
- $\tau : G \to G$ group automorphism, $\tau \circ \tau = \mathrm{id}_G$

Note. Any semisimple compact Lie group G has an involution σ such that $\sigma(t) = t^{-1}$ for all t in a *maximal* torus $T \subset G$. This σ is essentially unique (the *Chevalley involution*).

Example: $G = SU(n)$, $\sigma(g) = \bar{g}$.

$\Rightarrow \tau : \Omega(G) \to \Omega(G)$, $\tau(\gamma)(e^{i\theta}) = \sigma(\gamma(e^{-i\theta}))$

Obviously $\tau \circ \tau = \mathrm{id}_{\Omega G}$

- Symplectic form:

$$\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} < \xi(\theta), \frac{d\eta}{dt}(\theta) > d\theta$$

- Moment map for T action:

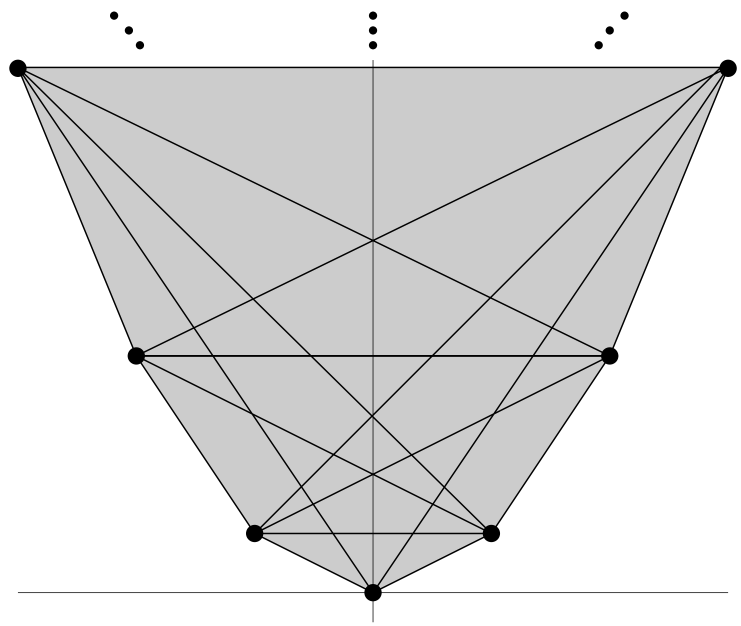
$$p(\gamma) = \frac{1}{2\pi} \int pr_{\mathfrak{t}}(\gamma^{-1}\frac{d\gamma}{dt})d\theta$$

- Moment map for S^1 action:

$$E(\gamma) = \frac{1}{4\pi} \int |(\gamma^{-1}\frac{d\gamma}{dt})|^2 d\theta$$

Theorem (Atiyah-Pressley 1983) The moment map $(E, p) : \Omega G \to \mathrm{Lie}(T) \oplus i\mathbb{R}$ is convex: in fact

$\Phi(\Omega G)) =$ convex hull $\Phi(\Omega(G)^{T \times S^1})$. Identify fixed point set of $T \times S^1$ action on ΩG: it consists of homomorphisms from S^1 to T (the integer lattice).



$\Phi(\Omega(SU(2))$

IV. DUISTERMAAT TYPE CONVEXITY

Would like Duistermaat convexity:

$$(?)\ \Phi(\Omega G)^\tau) = \Phi(\Omega(G))$$

Define an involution τ on ΩG as above: then τ compatible with the action $T \times S^1 \curvearrowright \Omega G$.

Extension of Duistermaat's convexity theorem to the based loop group:

(*)

$$\sigma(s) = s^{-1}$$

for all s in a maximal torus of G

Theorem. (Jeffrey-Mare 2010) If (*) holds, then the moment map $\Phi : \Omega G \to \mathrm{Lie}(T) \times i\mathbb{R}$ satisfies $\Phi(\Omega G) = \Phi\Big((\Omega G)^\tau\Big)$.

Example.
$G = SU(n)$, $\sigma(g) = \bar{g}\ \forall g \in SU(n)$.
Standard maximal torus T:
t is the diagonal matrix with eigenvalues $(z_1, \ldots, z_n)$ where
$|z_1| = \ldots = |z_n| = 1, z_1 \ldots z_n = 1$.
Check $\sigma(t) = t^{-1}\ \forall t \in T$.

$$\Rightarrow \Phi(\Omega G) = \Phi\Big((\Omega G)^\tau\Big).$$

0. $\Phi(\Omega G) \subset \mathrm{Lie}(T) \oplus i\mathbb{R}$ is a
convex polytope (Atiyah-Pressley)

1. $\Phi\left((\Omega G)^\tau\right) \subset \mathrm{Lie}(T) \oplus i\mathbb{R}$ is convex
(from Chuu-Lian Terng's convexity theorem for isoparametric submanifolds in Hilbert space)

2. If $\gamma \in \Omega G$ is such that $\Phi(\gamma)$ is a vertex of $\Phi(\Omega G)$, then $\Phi(\gamma) = \Phi(\tilde{\gamma})$, for some $\tilde{\gamma} \in \Omega(G)^\tau$.

0, 1, 2 $\Rightarrow$ $\Phi(\Omega G)) \subset \Phi\Big((\Omega G)^\tau\Big)$

V. Extension of Duistermaat's theorem to $H^*(\Omega G^\tau)$

Observation: if $K := \{g \in G \; : \; \sigma(g) = g\}$, then G/K is a Riemannian symmetric space.

Bott and Samelson 1958:

$$\dim H^{2q}(\Omega(G);\mathbb{Z}_2) = \dim H^q(\Omega(G/K);\mathbb{Z}_2) \; \forall q \geq 0$$

$$(\Omega G)^\tau = \{\gamma : S^1 \to G | \tau\big(\gamma(e^{i\theta})\big) = \gamma(e^{-i\theta})\}$$

This identifies $(\Omega G)^\tau$ with

$$\{\gamma : [0,1] \to G | \gamma(0) = e, \gamma(1) \in K\}$$

where $K = \{k \in G | \tau(k) = e\}$.

$\boxed{\Omega(G/K) \text{ is homotopy equivalent to } (\Omega G)^\tau}$

(follows from homotopy theory argument known since introduction of Borel construction, late 1950's)

Example: $G = SU(2)$, σ complex conjugation

$K = SO(2) \cong U(1)$

$G/K = S^2 = \mathbb{C}P^1$

Deduce CW decomposition of $(\Omega_{\text{alg}}G)^\tau = \bigsqcup_j C_j^\tau$.

Conclusion:

$$\dim H^{2q}\Big((\Omega G);\mathbb{Z}_2\Big) = \dim H^{2q}\Big(\Omega_{\text{alg}}G;\mathbb{Z}_2\Big)$$

$$= \#(2q\ \ - \text{dimensional cells } C_j)$$

$$= \#(q\ \ - \text{dimensional cells } C_j^\tau) = \dim H^q(\Omega_{\text{alg}}G)^\tau;\mathbb{Z}_2) = \dim H^q(\Omega(G)^\tau;\mathbb{Z}_2)$$

Theorem. (Jeffrey and Mare) *If σ is Chevalley involution of G and $K = G^\sigma$, then we have ring isomorphisms*

(*) $H^{2*}\left(\Omega G; \mathbb{Z}_2\right) \simeq H^*(\Omega(G/K); \mathbb{Z}_2)$

Main ideas of the proof.

- Identify $\Omega(G/K) = (\Omega G)^\tau$
- Replace ΩG by $\Omega_{\mathrm{alg}} G$ (see above).

Key point:

- τ leaves each cell of the CW decomposition invariant and acts on it as complex conjugation (see above)

Thus, $(\Omega_{\mathrm{alg}} G, \tau)$ is a *spherical conjugation complex* in the sense of Hausmann, Holm, and Puppe (2005). Then (*) is a direct application of results of [HHP] about spherical conjugation complexes with compatible torus actions.

VI. THE RINGS $H^*(\Omega(G/K);\mathbb{Z}_2)$

Examples:

$$G = SU(2):$$

$$K = SO(2) = S^1$$

$$G/K = S^2$$

$H^*(\Omega G;\mathbb{Z}_2) = \Lambda(\gamma_1, \gamma_2, \gamma_4, \gamma_8, \ldots)$ (where the degree of γ_j is $2j$).

Note: This is not valid without the assumption that the coefficient system is $\mathbb{Z}_2$.

$$H^*(\Omega S^2;\mathbb{Z}_2) = \Lambda(y_1, \delta_1, \delta_2, \delta_4, \delta_8, \ldots)$$

(where y_1 is a cohomology class of degree 1 and δ_j are cohomology classes of degree $2j$). The ring isomorphism

$$\mathcal{I} : H^*(\Omega S^2;\mathbb{Z}_2) \to H^*(\Omega G;\mathbb{Z}_2)$$

sends y_1 to γ_1 and δ_j to γ_{j+1} for all $j \geq 1$.

VII. Torus actions on moduli spaces of flat connections on 2-manifolds

•Moduli spaces of conjugacy classes of representations of the fundamental group of a 2-manifold into a compact Lie group G have a natural system of Hamiltonian flows (Goldman 1986).

•J-Weitsman (1992) observed that these flows are moment maps for a Hamiltonian torus action on an open dense set of moduli space. In the case $G = SU(2)$ the dimension of the torus that acts is half the dimension of the moduli space.

•These moduli spaces of conjugacy classes of representations are ordinarily singular, but for genus 2 and $G = SU(2)$ Narasimhan-Ramanan showed that the moduli space is smooth and isomorphic to $\mathbb{C}P^3$.

•In recent joint work with Nan-Kuo Ho, Khoa Dang Nguyen and Eugene Xia, we have concluded that the J-Weitsman torus actions can be used to identify the preimages of the interior of the moment polytope and its faces with the corresponding subsets of $\mathbb{C}P^3$.

•The torus action for the first and second copies of $U(1)$ are

$$u_1 \cdot (g_1, h_1, g_2, h_2) = (g_1 u_1, h_1, g_2, h_2)$$

$$u_2 \cdot (g_1, h_1, g_2, h_2 = (g_1, h_1, g_2 u_2, h_2)$$

The third torus action is

$$(g_1, h_1, g_2, h_2 \mapsto (e^{tX} g_1, h_1, e^{tY} g_2, h_2)$$

where X is the vector field

$$X(g_1, h_1, g_2, h_2) = h_2 h_1 - (h_2 h_1)^{-1}$$

and Y is the vector field

$$Y(g_1, h_1, g_2, h_2) = h_1 h_2 - (h_1 h_2)^{-1}$$

This means the only family of involutions compatible with the torus action is

$$\tau(g_1, h_1, g_2, h_2) = (e^{\lambda_3 X} g_1^s e^{\lambda_1 \xi_1}, h_1, e^{\lambda_3 Y} g_2^s e^{\lambda_2 \xi_2}, h_2)$$

Here $h_1 = e^{\xi_1}$, $h_2 = e^{\xi_2}$ and $\lambda_1, \lambda_2, \lambda_3$ are arbitrary real numbers.

Let s be a section of $M^0 \to \Delta^0$. Then

$$s(\mu([g_1, h_1, g_2, h_2]) = [g_1^s, h_1, g_2^s, h_2];$$

this defines g_1^s and g_2^s (which depend on the section s).

• We have also showed that the genus 2 moduli space is a conjugation space, by exhibiting an involution compatible with the torus action.

•The fixed point set of the involution τ is

$$\{(g_1, I, g_2, I)\}/G \cong (G \times G)/G$$

where I is the identity matrix.

Theorem:

•For all (g_1, h_1, g_2, h_2) whose images are on the boundary of the tetrahedron, $[h_1, h_2] = 1$.

•Also , if the images under the moment map are in the interior of the tetrahedron, then $[h_1, h_2] \neq 1$. It follows that these points do not represent flat $U(1)$ connections.

•It follows that the set of points whose image is in the interior of the tetrahedron is disjoint from the set of points whose image is in the boundary of the tetrahedron.

Hence the J-Weitsman Hamiltonian torus actions may be extended to the preimage of the interior of each face of the tetrahedron.

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