Jonathan Block [1] 2 schools of NCG: (of several)

1) Connes Operator algebras Today
2) Kontsevich Categories Tomorrow

} Friday

Beginning: Famous thm. of Gelfan'd, Naimark
$\exists$ equivalence of cats. btw.

$$\text{Cpct. top spaces} \longleftrightarrow \text{commutative } C^*\text{-algebras}$$

$$X \longmapsto C(X)$$
$$\hat{A} \longleftarrow\!\shortmid A$$

Lusztig (1971) Thesis proved Novikov's Theorem.

Signature theorem

$$H^{2k}(M,\mathbb{Q}) \times H^{2k}(M,\mathbb{Q}) \xrightarrow{\cup} H^{4k}(M,\mathbb{Q}) = \mathbb{Q}$$

sgn(M) homotopy invariant. and

Hirzebruch: $\operatorname{sgn}(M) = \langle L(M), [M] \rangle$

characteristic class in $H^{4k}(M,\mathbb{Q})$

$\operatorname{sgn}(M) = \operatorname{Index} D$ where D is an elliptic diff'l op.
mapping $\Omega^+(M) \longrightarrow \Omega^-(M)$, $\dim\ker D - \dim\operatorname{coker} D =: \operatorname{Ind} D$.

Q: Are there other CC's which are homotopy invariant? (rational CC's)

If M is simply-connected, answer is no.

Novikov Conjecture $f: M \longrightarrow B\pi$ Let $\alpha \in H^*(B\pi;\mathbb{Q})$

π is grp. form $\langle L(M) \cup f^*\alpha, [M] \rangle$
"higher signature"

Conjecture: these are homotopy invariants.

Lusztig proved the conjecture when $\pi = \mathbb{Z}^n$.

Let $T = V/\Lambda$, let $\Lambda = \mathbb{Z}^n$, T is a Torus and is a $K(\Lambda,1)$

Let $T^\vee = V^\vee/\Lambda^\vee$, $\Lambda^\vee = \{\xi \in V^\vee \mid \langle \xi, \lambda\rangle \in \mathbb{Z}\}$ $\forall \lambda$

$T^\vee \equiv$ Pontrjagin dual of Λ

= irreps. of Λ = flat unitary line bundles on T.

So, $\exists$ universal line bundle $\begin{matrix} P \\ \downarrow \\ T \times T^\vee \end{matrix}$ s.t. $P\big|_{T \times \xi} = \xi$

Consider $f: X \to T = K(\Lambda,1)$

form $(f \times 1): X \times T^\vee \to T \times T^\vee$, $\underset{\overset{\|}{\tilde{P}}}{(f\times 1)^* P} \to X \times T$

Can form the family of signature ops. $D_{\tilde{P}}$ on $X \times T^\vee$ over $\xi \in T^\vee$. $D_{\tilde{P}}$ is the signature of on X w/ "values in ξ"

$\operatorname{Ind} D_{\tilde{P}} = (\ker D_{\tilde{P}} - \operatorname{coker} D_{\tilde{P}})_\xi \in K^0(T^\vee)$ which you can show is a homotopy invariant.

Families Index thm of Atiyah-Singer says:

$\langle f^*\alpha \cup L(X), [X]\rangle$ (X is the same as M)

Myshchenko, Fomenko, Kasparov generalized this recall, $T = V/\Lambda = B\Lambda$, $T^\vee$ = Product of Λ

can take $\mathbb{C}[\Lambda] \to B(\ell^2(\Lambda))$ (by natural rep on left)

then close $\mathbb{C}[\Lambda]$ inside $B(\ell^2(\Lambda))$ to get $C^*(\Lambda)$.

Pontrjagin duality: $C^*(\Lambda) \cong C(\hat{\Lambda}) \cong C(T^\vee)$

Pontrjagin dual.

If want to apply Connes's idea to non-abelian fundamental grps then need to consider $C^*(\Gamma)$ and families index then should take place on "Spec $C^*\Gamma$."

If A is a C^*-alg. write $\hat{A}$:= space of all irred. $*$-reps on Hilb. spc.

Myshchenko, Fomenko, Kasparov generalized A-S families index then so that it takes place using $A = C^*(\pi_1 M)$ as a parameter spc.

Assume for simplicity, Γ is torsion free:

Aside:
For A a Banach algebra, define $K_*(A) := \pi_k(GL_\infty(A))$
$= \pi_k(BGL_\infty(A))$

then,

(Baum-Connes) conj: $K_*(B\Gamma) \xrightarrow{\sim} K_*(C^*\Gamma)$ isomorphism.

note: $\underbrace{K_*(B\Lambda) \longrightarrow K_*(C^*\Lambda)}_{\text{T-duality}} = K^*(T^\vee)$

What is the sense of treating $C^*\Gamma$ as functions on some space of irreducible modules over $C^*\Gamma$?

What does $C^*\Gamma$ look like? (It is a kind of moduli spc. of all unitary irreps of Γ)

Let α be a cardinal $\# \leq \aleph_0$ and let H_α be a Hilbert spc. of this cardinality. $I := \mathrm{Irrep}(\Gamma; H_\alpha)$ is a separable, metrizable, spc.

$\mathcal{U}(H_\alpha)$ acts on I. Quotient is $\widehat{C^*\Gamma} := I/\mathcal{U}(H_\alpha)$.

<u>dichotomy</u>: either,

Type 1 1) $\widehat{C^*\Gamma}^\alpha$ is very nice i.e. not too non-Hausdorff and can understand the structure of $C^*\Gamma$ by analyzing this $\widehat{C^*\Gamma}^\alpha$

Type 2 2) $I \xrightarrow{\text{proj}} I/\mathcal{U}(H_\alpha)$ $\nexists$ a Borel section of this projection.

many NC-spaces (in either context) arise from not wanting to take quotient (too early).

In both approaches think geometrically by looking at the category of modules

Jonathan Block | NCG 2 | Categorical approach.

In commutative geometry pts. $\longleftrightarrow$ irreducible modules

in NCG, things are more complicated.

Many examples: principally: quotients, deformations:

<u>Groupoids</u>: classify objects.

ex) groups, equiv. relation on a Set,

$$\begin{array}{c} R \subset X \times X \\ \downarrow\downarrow \\ X \end{array}$$

notation: $\mathcal{G} \underset{t}{\overset{s}{\rightrightarrows}} X$ objects $\qquad X/\mathcal{G}$ = quotient via equiv relation.

topological groupoids: (diff'l, alg) $\quad \mathcal{G} \underset{t}{\overset{s}{\rightrightarrows}} X \quad s, t$, have required structure (in Category)

$\mathcal{G} \times_X \mathcal{G} \to \mathcal{G}$ in Category

$\mathcal{G} \mapsto \mathcal{G}$ in Category

$\gamma \mapsto \gamma^{-1}$

Defn: in NC-world: can form $C^*(\mathcal{G})$ (analogous to the grp. alg.)

in Alg-geom: can form the stack $[X/\mathcal{G}]$ an

More examples of groupoids: G a grp acting on ~~top.~~ spc. X

have a map $G \times X \to X$: form groupoid:

$\mathcal{G}_1 = G \times X \underset{r}{\overset{s}{\rightrightarrows}}$ $\qquad s(g, x) = x$

$r(g, x) = gx$

so, $(g_1, x_1) \circ (g_2, x_2)$

if $x_1 = g_2 x_2$

$= (g_1 g_2, x_2)$

The quotient stack $[X/G] = X/_G$ should be thought of as a Morita-equivalence class of groupoids.

$\mathcal{G}_2 = G \times X \times H \rightrightarrows X \times H$

This isn't a groupoid, but $\mathcal{G}_1, \mathcal{G}_2$ have the same isom. classes of objects and the same isotropy.

From some point of view $\mathcal{G}_1$ and $\mathcal{G}_2$ are equivalent.

ex) Let X be mnfld., $\mathcal{U}_\alpha$ a cover of X by open sets.

Define a groupoid $\mathcal{U} := \coprod_{\alpha,\beta} (\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \overset{s}{\underset{r}{\rightrightarrows}} \coprod_\alpha \mathcal{U}_\alpha$

$s(x) = x \in \mathcal{U}_\beta \ , \ r(x) = x \in \mathcal{U}_\alpha$

also think of X as a groupoid trivially $X \overset{id}{\underset{id}{\rightrightarrows}} X$.

So $[X] \cong [\mathcal{U}]$

These represent the same stack.

ex) Let $(M^n, \mathcal{F}^q)$ be a foliated mnfld.

q = codimension

As an atlas, take a of q-dml submnfld T which intersects every leaf.

define groupoid $\mathcal{G} := \{$paths $\gamma: I \to M$ s.t. $\gamma(0) \in T, \gamma(1) \in T, \gamma \subset L$ a particular leaf$\}/\sim$

$\sim$ = homotopy in leaf rel. base points.

$\mathcal{G} \overset{r}{\underset{s}{\rightrightarrows}} T \qquad r\gamma = \gamma(1), \ s\gamma = \gamma(0)$ composition obvious

Statement $[\mathcal{G}_{T_1}] = [\mathcal{G}_{T_2}]$ for any T_1, T_2 satisfying the above condition.

form $C^*(\mathcal{G}_T)$. A leaf L of $\mathcal{F}$ gives an irreducible module of $C^*(\mathcal{G}_T)$.

$C^*(\mathcal{G}_{T_1}) \overset{M}{\simeq} C^*(\mathcal{G}_{T_2})$: i.e. are Morita Equivalent.

[what's intrinsic is A-mod not A itself.]

$A \rightsquigarrow M_n(A)$ A, B Morita equivalent if $\exists$ proj. A-module P s.t. $B \cong \mathrm{End}_A(P)$.

Moduli problems

A set of objects X you want to classify up to isomorphism.

try: $X/$isomorphism

- could be a nasty space
- could have isotropy:

keeps $X/\sim$ from being a fine moduli space

$$\forall \begin{array}{c} C \\ \downarrow \\ Y \end{array} \text{ get } \begin{array}{ccc} C & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ Y & \longrightarrow & M \end{array}$$

$\mathcal{C}$ ← univ. family

M ← moduli

Categories of modules

Take the whole cat. of objects and look for equivalences.

Kontsevich School of NCG

An NC space is a triangulated cat.

example: X

$$X \rightsquigarrow D^b(X)$$

$D^b(X)$ ← Cat. of certain $\mathcal{O}_X$ modules

Looking at D^b really only up to equivalence.

Can happen that $\exists$ X, Y not isom. varieties, yet $D^b(X) \cong D^b(Y)$.

when you have such an equivalence you are implicitly solving some moduli problem.

$$D^b(X) \longrightarrow D^b(Y)$$
$$\mathcal{O}_X \longrightarrow \Phi(\mathcal{O}_X)$$

ex) Mukai Duality $\quad$ X cmplx torus.

Y space of degree 0 line bundles on X

$= $ a dual torus

$$D^b(Y) \longleftrightarrow D^b(X).$$

ex) A beautiful deformation: Look at Heisenberg grp. $\quad x, y, z \in \mathbb{R}$

$$H = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{center } Z(H)$$

$$\begin{array}{c} C^*(H) \\ \downarrow \\ \mathbb{R} \end{array}$$

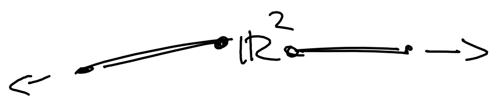
"Fourier transform at" the z coord.

$$\begin{array}{ccc} A_\xi & \longrightarrow & C^*(H) \\ & & \downarrow \\ & & \mathbb{R} \end{array}$$

$A_0 \cong C^*(\mathbb{R}^2) \cong C_0(\hat{\mathbb{R}}^2) = C_0(\mathbb{R}^2)$

$A_{\xi \neq 0} \cong$ Compact operators on $\mathbb{R}^2$

Is a deformation which describes $\mathbb{R}^2$ deforming into a pt.



[Bott Periodicity is a consequence of the deformation-invariance of this family]

$$\begin{array}{c} C(X)\otimes C^*(H) \\ \big\downarrow \\ \mathbb{R} \end{array}$$

over 0 get $X\times\mathbb{R}^2$

$\xi\neq 0$, $X\times\{pt\}$

Jonathan Block | NCG 3 $\qquad \Gamma$ is a torsion free group

Baum-Connes Conj.: $K_*(B\Gamma) \xrightarrow{\mu} K_*(C_r^*\Gamma)$

μ is an isomorphism

Description of μ the assembly map:

on $B\Gamma \times \operatorname{Spec} C_r^*\Gamma$ $\exists$ a line bundle, the Myschenko line bundle.

$$\begin{array}{c} \mathcal{V} = E\Gamma \times_\Gamma C_r^*\Gamma \\ \downarrow \\ B\Gamma \end{array} \quad \text{w/ fibers } C_r^*\Gamma \text{ as left modules.}$$

Can think of this as a "line bundle" over $B\Gamma \times \operatorname{Spec} C_r^*\Gamma$

Assume $B\Gamma$ is spin^c-mnfld. Then $K_*(B\Gamma) \cong K^{top *}(B\Gamma)$

$\mu : K^*(B\Gamma) \to K^*(C_r^*\Gamma)$

$$\begin{tikzcd} & B\Gamma \times \operatorname{Spec} C_r^*\Gamma \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ B\Gamma & & \operatorname{Spec}(C_r^*\Gamma) \end{tikzcd}$$

define μ via push-pull.

Remark: p_{2*} in K-thy is realized by taking the index of $\not{D}_{\mathcal{V}}$ along fibers of p_2.

Now to get something more algebraic and more subtle.

Want a description of coherent sheaves which is global diff'l geometric,
To be able to talk about cat of sheaves on NC spaces need to describe these objects in terms of global diff'l geometry.

Consider (for example on cmplx mnfld) the Dolbeaux alg.:

$$\mathcal{A} = (A^{0,\bullet}X, \bar\partial)$$

Thm: $\left\{\begin{matrix}\text{Hol VB's}\\ \text{on } X\end{matrix}\right\} \longleftrightarrow \left\{C^\infty \text{ VB's w/ flat } \bar\partial\text{-connections}\right\}$

that is: $\bar\nabla : C^\infty(X;E) \to A^{0,1}(X;E)$

$$\bar\nabla(sf) = \bar\nabla(s)f + s\bar\partial f$$

It's holomorphic if $\bar\nabla^2 = 0$.

Let $\mathcal{A} = (A^\bullet, d, c)$ curved DGA $\quad (d^2 = [c, \cdot])$

Define a DG-cat. $P_{\mathcal{A}}$ consisting of objects:

$(E^\bullet, \mathbb{E})$ $\quad$ $E^\bullet$ f.g., $\mathbb{Z}$-graded, proj. A^0-module

$\mathbb{E} : E^\bullet \otimes_{A^0} A^\bullet \circlearrowleft$ a total degree 1 homomorphism

satisfying $\quad \mathbb{E}(e\alpha) = \mathbb{E}(e)\alpha + (-1)^{|e|} e \otimes d\alpha$

curvature: $\quad \mathbb{E}^2(e) = -e \otimes c$

Morphisms: $\phi : (E^\bullet, \mathbb{E}) \to (F, \mathbb{F})$ of degree k

if $\phi : E^\bullet \otimes_{A^0} A \to F^\bullet \otimes_{A^0} A$ is of total degree k

$$d\varphi := \mathbb{F} \circ \varphi - (-1)^{|\varphi|} \varphi \circ \mathbb{E}, \quad (d^2 = 0)$$

$\therefore$ $P_{\mathcal{A}}$ becomes a DG-Cat.

Fact: $\mathrm{Ho}\,\mathcal{P}_A$ is triangulated.

Thm: If $A = (A^{0,\bullet}X, \bar\partial, 0)$, then $\mathrm{Ho}\,\mathcal{P}_A \cong D^b$(shvs w/ coherent cohomology X)

examples: If $(\mathcal{E}^\bullet, \delta)$ is a cmplx of holomorphic VB's

define $(E^\bullet, \mathbb{E})$: $E^\bullet = C^\infty(X, \mathcal{E}^\bullet)$, $\bar\partial_{\mathcal{E}} =: \mathbb{E}^1$ $\quad$ $\bar\partial$ is holomorphic

$\delta =: \mathbb{E}^0$

$\mathbb{E}^{k>1} = 0$

$$\mathbb{E}^{1\,2} = 0, \quad \mathbb{E}^{0\,2} = 0, \quad \mathbb{E}^0\mathbb{E}^1 + \mathbb{E}^1\mathbb{E}^0 = d\bar\partial_{\mathcal{E}} + \bar\partial_{\mathcal{E}} d = 0$$

exercise suppose $0 \to \mathcal{E}^0 \to \mathcal{E}^1 \to \mathcal{E}^2 \to 0$ SES of hol VB's

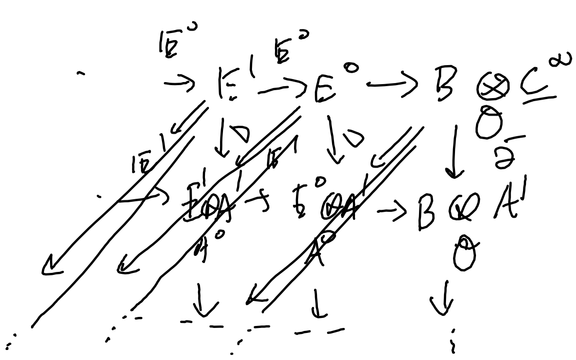
get cmplx of VB's

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E}^0 \arrow[r] \arrow[d] & \mathcal{E}^1 \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & \mathcal{E}^2 \arrow[r] & 0 \end{tikzcd}$$

a quasi isomorphism of VB's. In $\mathrm{Ho}\,\mathcal{P}_A$ these are isomorphic so construct the map backwards.

example: suppose $A \in \mathrm{Coh}\,X$. On cplx mnfld don't always have global resolutions by locally free sheaves. Take

$A \otimes_{\mathcal{O}} \underline{C}^\infty$ ← sheaf of C^∞-fnctns, will get resolution:



Thrm $B \in A^{0,2}X$, $\bar\partial B = 0$ B defines a class in $H^2(X,\mathcal{O}) \to H^2(X,\mathcal{O})$

B defines a topologically trivial Gerbe. $\downarrow$

$H^3(X,\mathbb{Z})$.

Let $A = (A^{0,\bullet}X, \bar\partial, B)$

$HoP_A \cong$ Derived Cat. of twisted coherent cohomology sheaves of weight 1.

Results related to this framework.

NC-Mukai Duality Take a torus $X = V/\Lambda$, V cplx vec. spc.

$B \in \Lambda^2 V^*$, $B = B^{2,0} + B^{1,1} + B^{0,2}$, B closed 2-form $B \in \Lambda^2 V^*$

$A_B = (A^{0,\bullet}X, \bar\partial, B) \leftarrow$ "gerby deformation of X"

$X^\vee = \bar V^\vee / \Lambda^\vee$ $C^\infty(X^\vee) \cong \mathcal{S}^*\Lambda$ define DGA $B^\bullet = \mathcal{S}\Lambda \otimes \Lambda^\bullet V_{1,0}$

w/ $V_{1,0} =$ (1-eigenspace of $V \otimes \mathbb{C}$)

$\bar\partial\lambda = 2\pi i \lambda\, D'(\lambda)$ $D' : V \to V_{1,0}$ proj.

Lemma $\mathcal{B} \cong (A^{0,\bullet}X^\vee, \bar\partial)$ Let $\sigma : \Lambda \times \Lambda \to U(1)$ be

$\sigma(\lambda_1,\lambda_2) = e^{2\pi i B(\lambda_1,\lambda_2)}$

form $\mathcal{B}_\sigma$: $[\lambda_1][\lambda_2] = \sigma(\lambda_1,\lambda_2)[\lambda_1+\lambda_2]$

the twisted algebra.

$\mathcal{B}_\sigma$ is highly non-commutative. $\bar\partial$ defined in same way.

Theorem: $\exists$ a deformed Poincaré object which implements an equivalence

$$HoP_{A_B} \xrightarrow{\sim} HoP_{\mathcal{B}_\sigma}$$

A deformed Fourier-Mukai transform.

suppose B is non-degenerate. Then the support of objects in P_{A_B} must be isotropic w.r.t. B.

Think of B_σ as Ψdo's These have to be isotropically supported and on the A_B-side these must be coisotropically supported } Gabber's Theorem