Caldararu: joint with D. Auribeau, D. Collare.

$X \rightarrow Y$ closed embedding of smooth varieties.

What can we say about

$$\text{Ext}_Y^*(i_*O_X, i_*O_X)$$

Describe:

- as vector spaces
- as algebras.

Open questions:

- $\text{Ext}_Y^*(i_*E, j_*F)$, $i: X \hookrightarrow Y$, $j: X' \hookrightarrow Y$

  $E, F$ vector bundles on $X$ and $X'$ respectively.

  There is a spectral sequence for these, but hard to work with.

  "Space of open string states between $i_*E$ and $j_*F"."

  - Particular case: $X \hookrightarrow X \times X$.

    $$\text{Ext}_X^{**}(\Delta_*O_X, \Delta_*O_X) =: \text{HH}^*(X)$$

    a) $\text{HH}^*(X) \cong \bigoplus_{p+q=*} H^0(X, \Lambda^q T_X)$ for $p, q \geq 0$

    (side question: is this true for schemes over char $p > 0$)

    b) $\text{HH}^*(X)$ is a graded commutative ring
\( \text{denote } \bigoplus_{p \geq k} H^p(X, \Lambda^p T_X) \text{ by } \hat{H}^p(X) \)

we have:
\( H^p(X) \cong \hat{H}^p(X) \) as rings but not via the HKR isomorphism, there are some corrections.

\[ \text{Hom}^*_X (O_X, O_X) \cong \text{Hom}^*_X (\Delta^* O_X, O_X) \]

Note: \( \Delta^* O_X \) is a Hopf algebra object in \( \mathcal{D}_b(X) \)
also note that the pullback above is derived:

\[ \Delta^* O_X \otimes \Delta^* O_X \xrightarrow{\Delta} \Delta^* O_X \]

Intuitively, \( \Delta^* O_X \) "should be" the structure sheaf
of \( LX \) which admits a map \( LX \to X \)

(loops)

If \( H \) is a Hopf algebra over a ring \( A \), assume \( H \) is commutative.

\[ Y = \text{Spec } H \xrightarrow{H \otimes H} \text{Spec } Y \xrightarrow{Y \times Y \to Y} \]

which matches fibers of \( Y \to X \) groups.
for a topological space, what is the "derived" (homotopy) fibered product of $X$ with itself as diagonal in $X \times X$? (LX)

\[
\begin{array}{c}
\Delta^* \Theta_{\Delta} \\
\Delta \times \Theta_{\Delta} \times X \\
\Delta \times X \times X
\end{array}
\]

\[\xymatrix{\Delta \ar[r] & \Theta_{\Delta} \ar[l] \\
X \ar[u] \ar[r] \ar[d] & X \times X \ar[u] \\
& \text{replace map by fibration} \ar[u]\}
\]

we have:
\[LX \to \text{Paths}(X) = \text{Hom}(I, X)\]
\[\begin{array}{c}
\begin{array}{c}
\text{(beginning, end)} \\
\text{of path}
\end{array}
\end{array}
\]

\[\begin{array}{c}
\begin{array}{c}
\Delta \ar[r] & X \times X \\
X \ar[u] \ar[r] \ar[d] & X \times X \ar[u] \\
\end{array}
\end{array}
\]

$LX$ is the homotopy fibered product.

Take $U = (\Delta^* \Theta_{\Delta})^\vee$ is another co-commutative Hopf algebra in $D^b(X)$.

\[\Rightarrow \text{ (Hilbert-Mumford) } U \cong U_{J_f} \to \text{universal envelope algebra of some dual algebra }\]

\[J_f = (T_X[-1], \alpha^+_{T_X[-1]}) \]

where $\alpha$ is the Atiyah class. For any $E \in D^b(X)$, $\alpha^+_{T_X[-1]} \in \operatorname{Ext}^1(E, T_X[-1] \otimes \mathcal{O}_X[-1])$. 

---

3
so in our case, it is a map at $T_x \mathfrak{g}$

this is a Lie algebra object.

every object corresponds to a representation

$\mathfrak{g} \rightarrow \text{trivial representation}$

$\text{Hom}^\mathfrak{g}_X (\mathfrak{g}, \mathfrak{g}) 
\ni \text{Hom}^\mathfrak{g}_X (\mathfrak{g}, \mathfrak{g})$ $\ni \text{Hom}^\mathfrak{g}_X (\mathfrak{g}, \mathfrak{g})$

Dyke's Theorem: $(\mathfrak{g}, \mathfrak{g})$ is the center of the universal enveloping algebra, $(\mathfrak{g}, \mathfrak{g})$ is an iso

which is a connection of the HKR iso, which is called the PBW isomorphism.

$\mathfrak{g} = \bigoplus \mathfrak{a}_i T_x \mathfrak{g} \mathfrak{i}$

$(\mathfrak{g}) = \text{Hom} (\mathfrak{g}, \mathfrak{g}) = \text{Hom} (\mathfrak{g}, \mathfrak{g})$

Back: $\text{Ext}^y (\mathfrak{g}, \mathfrak{g})$

$\text{Hom}^\mathfrak{g}_X (\mathfrak{g}, \mathfrak{g})$
Theorem: \( \Omega^0(\mathcal{O}_X) = \Lambda^0 N_{x/y}^x \)

This gives a spectral sequence with

\[ E^2 = \text{H}^9(\Lambda^0 N_{x/y}^x) \Rightarrow \text{Ext}^* \ldots \]

Theorem: (Ainbu, Calle) \( i^* i_* \mathcal{O}_X \approx S(N^*[1]) \)

If \( J \) a vector bundle on the \( \Omega \)-infinitesimal \n
\[ \text{whd of } X \times Y \text{ whose restriction to } x \text{ is } N_{x/y} \]

\( i^* i_* \mathcal{O}_X \) should be the "structure sheaf" on \n
the space of paths in \( Y \) starting and \n
ending at points of \( X \).

a groupoid \( X \), so \( i^* i_* \mathcal{O}_X \) should be a Hopf algebra.

*Conjecture*: \( (i^* i_* \mathcal{O}_X)^{\vee} = \mathcal{U}_Y / \mathcal{H}_U \)

\[ \mathcal{U}_Y = \text{T}_Y [c^{-1}] |_x \]

\[ \mathcal{H}_U = \text{T}_X [c^{-1}] \]
\[ \text{Hom}^*_Y (i_*O_X, j_*O_X) = \left( \underline{\text{H}^y} / \underline{\text{H}^y} \right)_Y \]

\[ \mathcal{E}_Y (i_*O_X, j_*O_X) \cong \left( \text{H}^0 \left( \text{H}^0 Y \right) \right) 
\quad \text{central} \]

\[ \text{sof} / \text{hsof} \]

\[ \bigoplus V^* \text{NC} \cdot i \cdot j \]

\textbf{Conjecture:} If \( Y \) has a \( \mathbb{Z}/2 \) action and fixed locus then \( \text{Ext}_Y (i_*O_X, j_*O_X) \overset{\cong}{\rightarrow} \bigoplus \text{H}^0 (X, \Lambda^N V) \)