1 Preface

I’ll describe work done with Davide Gaiotto, Greg Moore.

I will describe a new construction of hyperkähler spaces. They are torus fibrations (with degenerate fibers) over special Kähler manifolds.

The starting point for this construction is a rather complicated set of data. All these data would be summarized for physicists as “the infrared limit and BPS degeneracies of a 4-dimensional $\mathcal{N} = 2$ supersymmetric field theory.”

A crucial part of the data is a set of integer “invariants” which obey a rather complicated and exotic-looking wall-crossing formula, described by Kontsevich and Soibelman. They regarded it as a formula for wall-crossing of “generalized Donaldson-Thomas invariants”; we will reinterpret it as a formula for wall-crossing of “BPS degeneracies.” From the latter point of view, the construction of this hyperkähler manifold is physically very natural. It gives a purely physical/geometric perspective on why the wall-crossing formula should be true.

The idea is that to reduce the 4-dimensional theory to $\mathbb{R}^{2,1} \times S^1_R$. Once we do this, there are standard physical arguments which say that the “dimensionally reduced” physics is described by some hyperkähler manifold $\mathcal{M}$. If you try to calculate the metric on this manifold you find that it receives contributions from these particles going around $S^1$, weighed by $e^{-R|Z|}$. By looking carefully at these contributions one proves the WCF. Or conversely, the WCF allows you to construct this hyperkähler metric.

I’ll also describe an interesting set of examples where the moduli space you get is one which has been encountered before: moduli space of Higgs bundles (with singularities) over a Riemann surface...
2 SUSY gauge theory data

Begin with data:

- A complex manifold $\mathcal{B}$, of complex dimension $r$.
- A divisor $D \subset \mathcal{B}$. Let $\mathcal{B}' = \mathcal{B} \setminus D$.
- A local system of lattices $\Gamma$ over $\mathcal{B}'$, with antisymmetric pairings $\langle \cdot, \cdot \rangle$, such that $\Gamma_g = \Gamma / \text{rad} \langle \cdot, \cdot \rangle$ has rank $2r$.
- A “central charge” homomorphism $Z : \Gamma \to \mathbb{C}$, varying holomorphically over $\mathcal{B}$, constant on $\text{rad} \langle \cdot, \cdot \rangle$.
- Integer invariants $\Omega : \Gamma \to \mathbb{Z}$.

Subject to conditions:

- Transversality: $\langle dZ, dZ \rangle = 0$.
- Nondegeneracy: $\langle dZ, d\bar{Z} \rangle > 0$.
- Wall-crossing formula for $\Omega$: Define \textit{walls of marginal stability}, $W_\gamma \subset \mathcal{B}'$, by

$$W_\gamma = \{ u : \exists \gamma_1, \gamma_2 \in \Gamma_u, \text{linearly independent}, \gamma_1 + \gamma_2 = \gamma, Z(\gamma_1)/Z(\gamma_2) \in \mathbb{R}_+ \}.$$ 

Then $\Omega(\gamma; u)$ is constant over $\mathcal{B}' \setminus W_\gamma$. The jumping behavior of $\Omega(\gamma; u)$ is given by Kontsevich-Soibelman formula. Consider an algebraic symplectic torus $T_u := \Gamma_u^* \otimes_{\mathbb{Z}} \mathbb{C}^\times$. Each $\gamma$ canonically gives a function $X_\gamma : T \to \mathbb{C}^\times$, with $X_\gamma X_{\gamma'} = X_{\gamma + \gamma'}$. 
Define an element
\[ \mathcal{K}_\gamma : X_{\gamma'} \to X_{\gamma'}(1 - \sigma(\gamma)X_\gamma)\langle \gamma', \gamma \rangle. \]

(\(\sigma\) is a refinement of the antisymmetric pairing: \(\sigma(\gamma) = \pm 1\).)

Then choose a convex sector \(S \subset \mathbb{C}\) and form a product in clockwise order,
\[
A_S(u) = \left( \prod_{\gamma : Z(\gamma; u) \in S} \mathcal{K}_\gamma^{\Omega(\gamma; u)} \right) \in G
\]

\(A_S(u)\) is then invariant under deformation of \(u\) in which no \(Z(\gamma; u)\) enters or leaves \(S\). This is strong enough to determine \(\Omega(\gamma; u)\) from any \(\Omega(\gamma; u_0)\).

Simplest example of this is
\[
\mathcal{K}_{0,1}\mathcal{K}_{1,0} = \mathcal{K}_{1,0}\mathcal{K}_{1,1}\mathcal{K}_{0,1}.
\]

- Monodromies of \(\Gamma\) around \(D\) are compatible with \(\Omega(\gamma; u)\): around the component \(D_{\gamma_0}\), have
\[
\gamma' \to \gamma' + \sum_{\gamma \parallel \gamma_0} \Omega(\gamma; u) \langle \gamma, \gamma' \rangle \gamma.
\]

3 Example

A concrete example (pure \(SU(2)\) gauge theory): \(B\) is the complex plane, parameterized by \(u\). \(D\) consists of two points \(u = \pm \Lambda^2\), for some constant \(\Lambda > 0\). \(B\) is the base of a family of elliptic curves \(\Sigma_u\),
\[
\Sigma_u = \{ y^2 = (x^2 - u)^2 - \Lambda^4 \} \subset \mathbb{C}^2_{x,y},
\]
equipped with the one-form
\[
\lambda = \frac{1}{\pi \sqrt{2}} \frac{x^2}{y} dx,
\]
charge lattice $\Gamma_u = H_1(\Sigma_u, \mathbb{Z})$,

$$Z(\gamma; u) = \oint_{\gamma} \lambda.$$  

(Cuts run from $\pm \sqrt{u - \Lambda^2}$ to $\pm \sqrt{u + \Lambda^2}$.)

Draw the wall of marginal stability.

WCF in this case is

$$K_{2,-1}K_{0,1} = K_{0,1}K_{2,1}K_{4,1}K_{6,1} \cdots K_{2,0}^{-2} \cdots K_{8,-1}K_{6,-1}K_{4,-1}K_{2,-1}.$$  

4 The construction

So now assume we have all this $\mathcal{N} = 2$ gauge theory data.

Define $\mathcal{M}' \to \mathcal{B}'$ to be the total space of the real torus fibration $\Gamma^*_g \otimes_{\mathbb{Z}} (\mathbb{R}/2\pi \mathbb{Z})$. Any $\gamma \in \Gamma$ induces a coordinate function $\theta_\gamma$ on the fibers. (More precisely, twist the torus fibration by the set of quadratic refinements; let’s overlook this.)

Complete $\mathcal{M}'$ by including singular fibers appropriately, to a $C^\infty$ manifold (or sometimes orbifold) $\mathcal{M}$. We’ll construct a family of hyperkähler metrics $(\mathcal{M}, g)$ depending on a parameter $R \in \mathbb{R}_+$.  

Begin with the $R \to \infty$ limit. To describe it, use the standard complex structure in which

$$\pi R dZ_\gamma + i d\theta_\gamma$$  

is a $(1, 0)$ form. Then write a Kähler potential

$$K = R \langle Z, \bar{Z} \rangle.$$  

(Translation invariant along the fibers.) The torus fibers are flat, there are isometries which shift the angles; call this metric “semi-flat”, $g^{sf}$. It is actually hyperkähler.

At large $R$, the exact metric we are after is close to this one:

$$g = g^{sf} + O(e^{-R|Z|_{\text{min}}}).$$  

(4.3)
5 Twistorial construction

How are we to describe the exact metric efficiently? Basic idea: \( \mathcal{M} \) has three standard symplectic forms \( \omega_i \) and complex structures \( J_i \). In fact a \( \mathbb{C}P^1 \) worth; parameterize by \( \zeta \).

\[
J(\zeta) = \frac{i(-\zeta + \bar{\zeta}) J_1 - (\zeta + \bar{\zeta}) J_2 + (1 - |\zeta|^2) J_3}{1 + |\zeta|^2}.
\] (5.1)

For any \( \zeta \), \( (\mathcal{M}, J^{(\zeta)}) \) carries a holomorphic symplectic form,

\[
\varpi = -\frac{i}{2\zeta} \omega_+ + \omega_3 - \frac{i}{2} \zeta \omega_-.
\]

Knowing the holomorphic symplectic form for all \( \zeta \) is enough to reconstruct the hyperkähler metric. (A variant of twistor space construction.)

We’ll construct \( \varpi \) by giving “holomorphic Darboux coordinates” \( \mathcal{X}_\gamma(u, \theta; \zeta) \), for which

\[
\varpi = \frac{1}{4\pi^2 R} \frac{d\mathcal{X}_m}{\mathcal{X}_m} \wedge \frac{d\mathcal{X}_e}{\mathcal{X}_e}.
\] (5.2)

So suppose:

- \( \mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'} \).

- Each \( \mathcal{X}_\gamma(u, \theta; \zeta) \) is piecewise holomorphic in \( \zeta \) at any fixed \( (u, \theta) \in \mathcal{M} \).

- The \( \mathcal{X}_\gamma \) obey a reality condition,

\[
\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\zeta)}.
\] (5.3)

- All \( \mathcal{X}_\gamma \) are solutions to a single set of differential equations, of
the form
\[ \frac{\partial}{\partial u^i} \mathcal{X} = \left( \frac{1}{\zeta} \mathcal{A}_{u^i}^{(-1)} + \mathcal{A}_{u^i}^{(0)} \right) \mathcal{X}, \quad (5.4) \]
\[ \frac{\partial}{\partial \bar{u}^i} \mathcal{X} = \left( \mathcal{A}_{\bar{u}^i}^{(0)} + \zeta \mathcal{A}_{\bar{u}^i}^{(1)} \right) \mathcal{X}, \quad (5.5) \]

where the operators \( \mathcal{A}_{u^i}^{(n)} \), \( \mathcal{A}_{\bar{u}^i}^{(n)} \) are complex vertical vector fields on the torus fiber \( \mathcal{M}_u \), with the \( \mathcal{A}_{u^i}^{(-1)} \) linearly independent at every point, and similarly \( \mathcal{A}_{\bar{u}^i}^{(1)} \).

- \( \varpi \) has only a simple pole as \( \zeta \to 0, \infty \).
- The \( \varpi \) defined over different local patches of \( \mathcal{B} \) agree with one another, i.e. \( \varpi \) is globally defined.

Then, there exists an hyperkähler metric \( g \) such that \( \mathcal{X}_{\gamma}(u, \theta; \zeta) \) is holomorphic on \( (\mathcal{M}, J^{(\zeta)}) \) for fixed \( \zeta \), and \( \varpi \) is the holomorphic symplectic form.

### 6 Semi-flat twistor construction

The metric \( g^{sf} \) arises from the choice
\[ \mathcal{X}_{\gamma}^{sf}(\zeta) := \exp \left[ \pi R \zeta^{-1} Z(\gamma; u) + i \theta_{\gamma} + \pi R \zeta \bar{Z}(\gamma; u) \right]. \quad (6.1) \]

Note the essential singularity at \( \zeta = 0 \). Characteristic of solution to a differential equation with irregular singularities.

### 7 Riemann-Hilbert problem

Proposal: for each fixed \( u \), obtain the desired functions \( \mathcal{X}_{\gamma}(u; \zeta) \) as the solution to an infinite-dimensional "Riemann-Hilbert problem."

We require:
• \( \mathcal{X}_\gamma \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'} \),

• Each \( \mathcal{X}_\gamma^{-1}(-1/\bar{\zeta}) = \mathcal{X}_{\gamma}(\zeta) \),

• Each \( \mathcal{X}_\gamma \) is piecewise-analytic in \( \zeta \),

• The collection \( \mathcal{X} = \{ \mathcal{X}_\gamma \} \) jumps by \( K_\gamma^{\mathcal{X}(\gamma;u)} \) along the line

\[ \mathcal{L}_\gamma = \{ \zeta : Z(\gamma; u)/\zeta \in \mathbb{R} \} \]

• Each \( \mathcal{X}_\gamma(\mathcal{X}_{\gamma}^{-1}) \) is finite as \( \zeta \to 0, \infty \).

If they exist, the \( \mathcal{X}_\gamma \) obey all of our conditions for the twistor construction. Argue that the desired \( \mathcal{X}_\gamma \) indeed exist, for sufficiently large \( R \), by formulating the solution in terms of an integral equation:

\[
\mathcal{X}_\gamma(\zeta) = \mathcal{X}_{\gamma}^{\text{sf}}(\zeta) \exp \left[ -\frac{1}{4\pi i} \sum_{\gamma'} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \times \int_{\mathcal{L}_{\gamma, \gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta + \zeta'}{\zeta' - \zeta} \log(1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta')) \right]
\]

But, our jump conditions will change when the BPS spectrum changes. So the \( \mathcal{X}_\gamma \) have no reason to be continuous unless the W CF is satisfied! Said differently, the hyperkähler metric is not continuous unless the WCF is satisfied.

8 Comments about \( g \)

Does the solution exist for small \( R \)? Physically, we expect yes (certainly \( \mathcal{M} \) does). But no mathematical proof. It might only work when the \( \Omega(\gamma; u) \) come from a physical theory!

Behavior at the singular divisor \( D \): one expects physically that the singularities in \( g^{\text{sf}} \) will be resolved or at least reduced in \( g \). Can
check this directly for the simplest kind of singularity, where a single $Z(\gamma; u) \to 0$ with $\Omega(\gamma; u) = 1$. If $\gamma$ is $q$ times a primitive vector, we get an $A_{q-1}$ singularity in $\mathcal{M}$. (So for $q = 1$ there is no singularity: this example is “Ooguri-Vafa” space.) Behavior at loci where components of $D$ meet remains to be understood, probably interesting.

9 Higgs bundles

An important special case of our construction: take $\mathcal{M}$ to be the moduli space of solutions of Hitchin equations on a curve $C$ (with ramification).

Consider a $SU(2)$-connection $D$ on a bundle $V$ over $C$, and $\varphi \in \Omega^{1,0}(\text{End} V)$. Hitchin’s equations:

\[ \bar{\partial}_D \varphi = 0 \]
\[ F_D = R^2[\varphi, \varphi^*] \]

$\mathcal{M}$ is the space of solutions, modulo gauge equivalence.

To be precise, consider case where $D$ and $\varphi$ have simple poles at finitely many points $z_i$, with fixed residues.

Recall that $\mathcal{M}_{\zeta=0}$ is moduli space of Higgs bundles: pairs $(E, \varphi)$ where

- $E$ is a holomorphic rank $N$ vector bundle over $C$
- $\varphi \in H^0(\text{End} E \otimes K)$

Given $\varphi$ define the spectral curve to be the double covering given by the 2 eigenvalues of $\varphi$:

\[ \Sigma = (z, x) : \det(x - \varphi(z)) = 0 \subset T^*C \]

In our case $\det(x - \varphi(z)) = x^2 - u$ where $u$ is a quadratic differential on $C$, with double poles at the $z_i$. 
This moduli space will arise from our construction. What is the data?

• \( \mathcal{B} \) is the space of holomorphic quadratic differentials \( u \) on \( C \), with double poles at each \( z_i \), fixed residues.

• Define \( \Sigma_u = \{(z, x) : x^2 - u = 0\} \subset T^*C \). Then

\[
\Gamma_u = \{ \gamma \in H_1(\Sigma_u, \mathbb{Z}) : \sigma \gamma = -\gamma \}.
\]

\( \langle , \rangle \) is the intersection pairing.

• There is a canonical 1-form

\[
\lambda = x dz
\]

on \( \Sigma \); the central charge map is

\[
Z(\gamma) = \oint_\gamma \lambda.
\]

• Given \( u \) and an angle \( \vartheta \) there is a real foliation on \( C \): define the leaves ("WKB curves") to be ones for which \( \lambda \in e^{i\vartheta} \mathbb{R} \). For generic \( \vartheta \) all leaves meet a singular point. The integers \( \Omega(\gamma; u) \) are counting how many leaves of finite extent appear as \( \vartheta \) is varied. Two types. Saddle connections: define \( \gamma \) to be the difference of two lifts to \( \Sigma \), then these contribute \( \Omega(\gamma; u) = +1 \). Closed loops: define \( \gamma \) to be the difference of two lifts to \( \Sigma \), then these contribute \( \Omega(\gamma; u) = -2 \).

Physical interpretation: 2 M5-branes wrapped on \( C \), intersecting some other M5-branes at the points \( z_i \). This gives \( \mathcal{N} = 2 \) theories in 4 dimensions. For example, \( N_f = 4 \) Seiberg-Witten theory corresponds to \( C = \mathbb{CP}^1 \) with 4 regular singularities: the residues of the Higgs field determine the masses.
Can also consider irregular singularities, e.g. these arise if we take $N_f < 4$. The simple example we started with corresponds to 2 irregular singularities of the weakest possible kind (“Steinberg variety”).

10 Darboux coordinates

Now to see how our description of the metric goes, need to construct the $\mathcal{X}_\gamma$. What does $\mathcal{M}_\zeta$ look like for $\zeta \in \mathbb{C}^\times$? Consider the (complex) connection:

$$\nabla_z = R\zeta^{-1}\varphi_z + D_z$$
$$\nabla_{\bar{z}} = R\zeta \bar{\varphi}_{\bar{z}} + D_{\bar{z}}$$

Hitchin’s equations imply $\nabla$ is flat. This identifies $\mathcal{M}_\zeta$ with a moduli space of flat connections.

So holomorphic functions on $\mathcal{M}_\zeta$ are gauge-invariant quantities built out of $\nabla$. $\mathcal{X}_\gamma$ on $\mathcal{M}$ will be obtained this way.

Construct *canonical triangulation* $T(\vartheta, u)$ from the WKB foliation. One branch point in each face. Then to any quadrilateral associate a homology cycle:

![Diagram](image)

Figure 1: The construction of $\gamma_E \in H_1(\Sigma, \mathbb{Z})$. 
Fock-Goncharov defined some coordinates on moduli spaces of flat connections, depending on:

- A triangulation of $C$. Take $T(\vartheta = \arg \zeta, u)$.
- A monodromy eigenvector at each singularity. Take the one which is exponentially smaller in norm along the edges.

Fock-Goncharov define the coordinate by

$$X_\gamma := -\frac{(s_1 \wedge s_2)(s_3 \wedge s_4)}{(s_2 \wedge s_3)(s_4 \wedge s_1)}.$$

Now want to check $X_\gamma$ have the properties we claimed.

11 Jumps

As $\vartheta$ varies the foliation jumps in a specific way. The simplest jump comes from a saddle connection and gives a flip of the triangulation. Fock-Goncharov already studied the transformation of the triangulation under flips: it gives exactly the jump $K_{\gamma_{BPS}}$ we needed for the WCF with $\Omega(\gamma) = 1$.

There is also a more intricate jump coming from the appearance of a family of closed curves. This one was not studied by Fock-
Goncharov. It gives a jump of the $\mathcal{X}_\gamma$ by $\mathcal{K}^{-2}_{\gamma BPS}$, exactly as we needed with $\Omega(\gamma) = -2$.

Show the animations.

12 Asymptotics

WKB approximation verifies the asymptotics of the coordinates $\mathcal{X}_\gamma(\zeta)$ as $\zeta \to 0$. Namely, in this approximation evaluating the parallel transport just reduces to integrating the Higgs piece $\varphi/\zeta$. That’s
not exact, but you can neglect the error, because the contamination is exponentially smaller than the piece you are keeping.

One obtains as $\zeta \to 0$

$$\mathcal{X}_\gamma \sim c_\gamma \exp \left[ \frac{1}{\pi} \oint \lambda \right].$$

This is what we needed in order to “cap off” the twistor space.