1) when are two complexes \( q. i.s.o.m \)?

a) \( H^k(C_1) \neq H^k(C_2) \), then not \( q. i.s.o.m \).

b) even if \( H^k(C_1) \cong H^k(C_2) \) still don’t know

\[
\begin{align*}
C_1 & : 0 \to A^{\otimes 2} \xrightarrow{(x,y)} A \to 0 \\
C_2 & : 0 \to A \xrightarrow{\partial} k \to 0 \\
k & = A/(x,y)
\end{align*}
\]

note: kernel is the submodule generated by \((y,-x)\), it is \( \cong A \)

Note: hard to prove \( C_1, C_2 \) not \( q. i.s.o.m \) directly from the definition so the method is to construct an invariant \( A \) for \( q. i.s.o.m \) which differs between \( C_1 \) and \( C_2 \).

Claim: every complex \( C^* \) which has only two non-zero adjacent homology groups is completely determined by \( H^0(C^*) \) and \( H^1(C^*) \) and an extension class \( e \in \text{Ext}^2(H^0, H^1) \)

how do you get the extension class from the complexes above?

take \( M \to N \) and expand this to the exact sequence

\[
\begin{array}{c}
0 \to \ker \to M \to N \to \coker \to 0 \\
\in \text{Ext}^2(\coker, \ker)
\end{array}
\]

For the complex \( C_1 \) above, get

\[
\begin{align*}
0 & \to A \\
A & \to A^{\otimes 2} \to A \\
& \to A/(x,y) \to 0
\end{align*}
\]

which is a Koszul resolution \( F \) of \( A/(x,y) \) over \( A \)

Geometrically, think of \( A \) as \( \mathbb{A}^2 \), \( A/(x,y) \to O_0 \) where \( 0 \) is origin in \( \mathbb{A}^2 \)

We look for \( e \in \text{Ext}^2(O_0, O_0^\vee) \cong \text{Ext}^0_{\mathbb{A}^2}(O_0^\vee, O_0)^\vee \cong k \\
\) some duality

in general, if \( C^* \) has only \( H^* \) and \( H^{-\cdot} \) can always map

\[
\begin{array}{c}
H^*(C^*) \xrightarrow{\delta} C^* \\
H^{-\cdot}(C^*) \xrightarrow{\delta} C^*
\end{array}
\]

gives a map in derived category

\[
\begin{array}{c}
H^*(C^*)[0] \to H^{-\cdot}(C^*)[1] \\
H^*(C^*)[1] \to H^{-\cdot}(C^*)[2]
\end{array}
\]

This method does not generalize to complexes with more homology. Instead...

Fact: can regard \( A \) as an \( A_\infty \)-algebra and \( C^*_1, C^*_2 \) as \( A_\infty \)-modules \( M \). Then \( C^*_1 \cong C^*_2 \) if they are isomorphic as \( A_\infty \)-modules.
Kontsevich's philosophy: For homological purposes, schemes can be regarded as "affine," but the rings involved are $A_\infty$.

**Theorem:** (Bondal–van den Bergh; Toën–Vezzosi)

Let $X$ be a complex mfd, then $X$ is algebraic if the derived category $D^b(X)$ admits a split generator.

**Fact:** If $E$ is a split-generator for $D^b(X)$, then $D^b(X)$ can be recovered from the $A_\infty$-algebra $\text{Ext}^*(E,E)$.

Note this is a weaker notion. Very few spaces admit a true generator.

So we have loose correspondence:
- affine variety $\rightarrow$ ring
- alg. variety $\rightarrow$ $A_\infty$-ring

This correspondence is not tight. Some $A_\infty$-rings don't come from any variety, and many varieties have same $D^b(X)$, i.e. Fourier–Mukai transforms.

**Original motivation:** (Stasheff): (See Keller's introduction to $A_\infty$ algebras)

If $(X,\ast)$ is a pointed space, consider $\Omega(X)$ the space of loops at $\ast$. Want to express that $\Omega(X)$ is a group up to homotopy.

Composition of loops not associative:

\[
\begin{array}{c c c}
A & B & C \\
B & A & C \\
C & A & B \\
\end{array}
\]

\[
(A \ast B) \ast C \\
A \ast (B \ast C)
\]

Composition not enough: there is some homotopy between the two, but it's not associative up to same prescribed homotopy.

Well behaved with respect to compositions in the sense that there is a higher homotopy filing the associator pentagon, and higher and higher...

And Stasheff's result is that a space is equivalent to a loop space precisely when it admits such an $A_\infty$ structure.

An $A_\infty$-algebra is an algebraic analogue of this; want to relax associativity.
Almost definition: An $A_{\infty}$ algebra is a graded vector space $A$ together with

\[ m_k : A^k \rightarrow A, \quad k \geq 1 \]

satisfying an infinite sequence of equations

\[ \sum_{\sigma} \pm m_k (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = 0 \]

where $\sigma$ satisfies some higher equations

Call $m_1 = d$, $m_2 = \cdot$, then $\text{Assoc}_{1,2,3}$ imply

1. $d^2 = 0$
2. $d(xy) = xdy - d(x)y = 0$
3. $(xy)^2 - x(yz) = d m_3 (x, y, z) \pm m_3 (dx, dy, dz) \pm m_3 (x, y, dz)$

If $m_3 = 0$, $\cdot$ is associative,
\[ \Rightarrow \text{gives an associative mult on } H^*(A, d) \]
from $m_2 : A \otimes A \rightarrow A$

Ex: non-assoc. algebras, $m_2 \neq 0$, all others 0
- dg algebras: $m_k = 0$, $k > 2$
- example: forms on a manifold admit finite dimensional minimal models

Two facts: complexes of vector spaces are formal

\[ C^* = \bigoplus C^i C^* [-i] \]

Then: (Deligne, Griffiths, Morgan, Sullivan)
let $X$ be a Kähler manifold, then $X$ is formal

Other examples: Let $A$ be an assoc. algebra and
let $\alpha$ be an assoc. $k$-cocycle

\[ \alpha : A^k \rightarrow A \]

then can form an $A_{\infty}$ algebra $A_{\alpha}$ setting $m_2 = \cdot$

Aside on deformations:
given assoc. alg. $A$, get a new algebra which depends on $	riangleright$

\[ X \ast Y = xy + \delta X c_2(xy) + \delta^2 c_3(xy) + \cdots \]

now associative up to $\delta^2 \iff c_2$ satisfies

\[ Xc_2(xy) - c_2(xy, z) + c_2(xy, z) = 0 \]

If $s : A \to A$, $c_2(x, y) = xs(y) - s(xy) + s(x)y$

consider these to be trivial, when you quotient out, get Hopf $(A)$

Comment on flat vs. curved

$\triangleright$ if we allow $m_k$'s for $k > 0$, called a curved $A_\omega$-algebras

$\begin{align*}
  m_0 & : k = A \\
  m_0 & : 0 = W, m_2 = 1, m_2 = 0
\end{align*}$

$\begin{align*}
  \text{Assoc}_0 & : dW = 0 \\
  \text{Assoc}_1 & : d^2(x) = xW + WX = 0 \\
  \quad d^2 & \neq 0 \text{ in general}
\end{align*}$

$\begin{align*}
  \text{Assoc}_2 & : \text{messy...}
\end{align*}$

$\begin{align*}
  \text{Examples of curved } A_\omega \text{-alg.} & \\
  & A \text{ assoc. alg., } W \in \mathbb{Z}(A) \\
  & \text{set } m_0(1) = W, m_2 = 0, \text{ all other } 0
\end{align*}$

Free coalgebras:

$\begin{align*}
  V \mapsto TV = \text{tensor algebra on } V \\
  = \bigoplus_{k=0} V^\otimes k
\end{align*}$

free coalgebra $C(V) = \bigoplus_{k=1} V^\otimes k$

(note this def.

is different than

Keller's paper because

it includes curved algebras)

$\begin{align*}
  \Delta(x_1 \cdots x_n) = \sum_{k=0} \delta(x_1 \cdots x_k) \otimes (x_{k+1} \cdots x_n)
\end{align*}$

Note: this is not a Hopf algebra. There are other products/coproducts called shuffle prod/cod space.
There is a natural isomorphism

\[ \text{Hom}_{\text{Vec}}(V, TV) \cong \text{Der}(TV, TV) \]

Claim: dual theorem holds!

\[ \text{Hom}_{\text{Vec}}(TV, V) \cong \text{CoDer}(TV, TV) \]

\( b \) is coderiv. if \( \Delta b = (b \circ 1 + 1 \circ b) \Delta \)

Check this by forming map on homogeneous pieces

\[ \begin{align*}
Y \quad &\mapsto (Y11) + (1Y1) + (11Y) \\
\end{align*} \]

Theorem: An \( A_\infty \)-structure on \( V \) is the same thing as a square 0 coderivation on \( CV \)

\[ m_0 \quad \mapsto \quad m_1 \quad \mapsto \quad m_2 \quad \mapsto \quad m_3 \]

can use these to form diagrams of composition and express relations

An analogy:

\( \text{Der}'s \) of \( A_\infty \) algebras

\( \text{Hom}(CV, V) \cong \text{Coder}(CV, CV) \)

Huge manifold

\( T_A \& D^2 = 0^3 \)

\( D_0 \) specific \( A_\infty \) alg. \( A \)

\( \text{locus } D^2 = 0 \)

A 1st order derivation of \( D_0 \) is of the form \( D_0 + fD_1 \)

\( \Rightarrow \) want to satisfy \( (D_0 + fD_1)^2 = 0 \) to first order

\( D_0^2 + f(D_0D_1 + D_1D_0) + \ldots = 0 \)

\( \Rightarrow \) the tangent space to the def. space of an \( A_\infty \)-alg. \( A \) rep.

\( \left\{ D_1 \in \text{Coder}(CA, CA) \mid [D_0, D_1] = 0 \right\} \)

writing the Hochschild complex
Let $X$ be a scheme $\mathcal{Coh}(X)$ abelian category. Complexes $(\mathcal{A}) = \text{cat}$, whose objects are complexes related construction $dg(\mathcal{A})$

objects: complexes $\hom^i(\mathcal{C}^\bullet, D^\bullet) = \mathcal{T} \hom(\mathcal{C}^\bullet, D^\bullet)$

\[ d : \hom^i(\mathcal{C}^\bullet, D^\bullet) \rightarrow \hom^{i+1}(\mathcal{C}^\bullet, D^\bullet) \]

\[ d(F^\bullet) = d_F F^\bullet \oplus F^{\geq 1} \]

This is a $dg$-category enriched over complexes $\hom^0(\hom^0(\mathcal{C}^\bullet, D^\bullet)) = \text{h.top. classes of chain maps}$

g category S collection of morph.
localization has the universal property

\[ C \xrightarrow{F} D \xrightarrow{F} \text{if } F(x) \text{ an iso, for all } x \in S, \text{then } \exists! \text{ (up to nat. isomorph?) } \]

\[ A \triangleleft X_1 \triangleleft X_2 \text{ } \cdots \cdots Y_n \triangleleft X_{n-1} \triangleleft X_n \triangleleft B \]

relations are 2-morphisms

given $dg(\mathcal{A}) \rightarrow H_0(\mathcal{A})$
morphisms are $H_0(\text{morph. in } dg(\mathcal{A}))$

(Note: there is a theory of q.iso.s in $dg(\mathcal{A})$

For category $H_0(\mathcal{A})$, let $S \in \text{equid. isomorphisms}$

\[ D(\mathcal{A}) = S^{-1} H_0(\mathcal{A}) \]

the "derived category"

Theorem: if $\mathcal{C}^\bullet, D^\bullet$ are... complexes of injectives then a map $\mathcal{C}^\bullet \rightarrow \mathcal{D}^\bullet$ is a q.iso. if it is invertible up to homotopy

Thm: $D(\mathcal{A})$ is equivalent to the $H_0(\text{Inj}(\mathcal{A}))$
Thm: $D(A)$ is equivalent to the $\text{Ho}(\text{Inj}(A))$

We want to consider $DG(A) = \text{dg}(\text{Inj}(A))$.

Nowadays people have realized that working with the dg category is better than passing to homotopy.

So $DG(A)$ is a dg-category, with $H^0O_G(A) = D(A)$, i.e. $DG(A)$ is an "enhancement".

---

**Homological Perturbation Lemma**

Introduce a $\mathbb{Z}$ or $\mathbb{Z}/2$ grading in our $A_\infty$ algebras $A$.

$m_k : A^\otimes k \rightarrow A[2-k]$ satisfying $\text{Assc}m$ with proper signs.

(Following *p. 41 onwards*)

Now let $A$ be a $\mathbb{Z}$ graded flat $A_\infty$-algebra, $d = m_1$.

We want to write $A^\wedge = B^\wedge \otimes ?$.

we can think of this as having $\Pi : A \rightarrow A$ with $\Pi^2 = \Pi$.

Assume $\exists \ H : A \rightarrow A[1]$ s.t.

$1 - \Pi = dH + Hd$

Now let $B^\wedge = \text{im} \Pi$, we'll define an $A_\infty$ alg. str. on $B^\wedge$.

---

Homological perturbation lemma:

Have $A$ a $\mathbb{Z}$ graded, $d : A \rightarrow A[1]$, $d^2 = 0$.

$m_k : A^\otimes k \rightarrow A[2-k]$.

$\Pi : A \rightarrow A$, $\Pi d + d \Pi = 0$, $\Pi^2 = \Pi$.

$H : A \rightarrow A[1]$, $1 - \Pi = dH + Hd$. $B = \text{im} \Pi$.

$A_\infty$ struct. on $B$.
$$d_B = d_A|_B,$$
$$m_B^i = p_i m_A^i (i \theta i) \quad \iff \quad \bigg\uparrow_i \bigg\downarrow_i \quad \text{all dots are ops in } A,$$
$$m_B^i \bigg\uparrow_i \bigg\downarrow_i \quad \text{leaves are labelled by } i, \quad \text{root is labelled by } i.'$$

$$m_B^n = \sum_T \pm m^n_T,$$
where \( T \) is any planar oriented tree with leaves,
valency of each internal vertex \( \geq 3 \)

$$m^n_T : m_A^n \rightarrow B[2-n]$$

label internal vertices by \( m^i_k \),
according to valency

leaves \( \big\uparrow i \)
root \( \big\downarrow p \)
internal edge \( \big\uparrow H \)

**Theorem**: \( d_B, m_B^i \) form an \( A \)-algebra structure on \( B \)

We want to show that

\[ \sum m_B^i \bigg\uparrow_i m_B^i \bigg\downarrow_i = 0 \]

ie.

\[ \sum \bigg\uparrow_i = 0 \]

Note we can realize this as the sum over all trees with:
- leaves labelled \( i \)
- root labelled \( p \)
- exactly one internal edge labelled \( H \),
all others labelled \( \Pi \)

Plus all the terms that have \( m_B^i \) acting on either the root or one of the leaves

So we must have

\[ 0 = \sum_T \left( \text{as before with one internal edge labelled } \Pi \right) + \sum_T \left( \text{w/ a dot on } \Pi \text{ a leaf or the root} \right) \]

Once again using the notation

\[ Y = m_A^i, \quad \phi = H, \quad \Psi = \Pi \]

and \( 1 - \Pi = HH + \Pi \Pi \) means

\[ \phi \Psi + \phi H + \phi d = 1 + \Psi \]
Idea: putting one σ on each leaf and the root of the tree gives sum over all decompositions into two trees. E.g.

\[ \chi = \chi \pm \chi \pm \chi = \pm \chi \pm \chi \]

Now define

\[ \hat{m}^{B}_{n} = \sum_{\text{black dot added on each edge}} \left( \pm \right) \]

E.g.

\[ \hat{m}^{B}_{3} = \chi \pm \chi \pm \chi \pm \chi \pm \chi \pm \chi \pm \chi \]

\[ \pm \chi \pm \ldots \]

I.e. each internal edge, labelled with white dot, \( q = \mathbf{1} \), now counts as two edges, above and below.

So

\[ \hat{m}^{B} = \sum_{\text{black dot on leaf or root}} \left( \pm \right) + \sum_{\text{white dot replace}} \left( \pm \right) \]

Now we will compute \( \hat{m}^{B}_{n} \) a different way, and get

We have

\[ \hat{m}^{B}_{n} = \sum_{\text{label each vertex with black dot}} \left( \pm \right) = \sum_{\text{erase white dot}} \left( \pm \right) \]

E.g.

\[ \hat{m}^{B}_{3} : \]

Another grouping...
For instance

\[ a \pm b = a \mp b \]
\[ = \begin{array}{c}
  \pm \\
  \pm \\
\end{array} 
\]
\[ = \text{Assoc}_x \]

\[ \begin{align*}
  \text{space} & \quad \text{DG category} \quad \text{DG}(X) \\
  & \quad \rightarrow \quad \text{HPL} \\
  & \quad \rightarrow \quad \text{A}_\infty \text{ category} \quad \text{A}_\infty(X) \\
  & \quad \text{(easy to construct)} \\
  & \quad \text{small, so } m_i \text{ big} \\
\end{align*} \]

\[ \text{note: } \text{A}_\infty \text{ category} \]
\[ m_n : \text{sequence} \quad (A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_{n+1}) \]
\[ \quad \quad \rightarrow \quad (A_1 \xrightarrow{m_n(f_1, \ldots, f_n)} A_{n+1}) \]

this seems like a lot of information, but we have

Thus, if \( E \) is a split generator of \( \text{D}^b(X) \), then the \( \text{A}_\infty \) algebra \( \text{End}(E) \) carries all the information of the \( \text{A}_\infty \) category \( \text{A}_\infty(X) \)

- generator: generates everything w/ cones and shifts
- split generator: cones, shifts, and direct summands

Examples: on \( \mathbb{P}^n \), \( \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n) \) generate (Beilinson)

On an elliptic curve, no finite set of generators

\[ \begin{align*}
  \text{pf uses map} & \quad \text{D}^b(X) \rightarrow \text{K}^*(X) \\
  \text{} & \quad C^* \rightarrow \sum (-1)^i [C^i] \\
  \text{} & \quad \text{exercise} \quad \sum (-1)^i [H^i(C^*)] \\
\end{align*} \]

Also have property

\[ \text{A}_\rightarrow \text{R}_\rightarrow \]
\[ A^2 - B^2 + C^2 = 0 \]

Special case of this is
\[ [A^*B^*] = -[A] \]

Now on an elliptic curve \( D^b(x) \) is not finitely generated because \( K^*(x) \) has a class corresponding to each point (uncountably many)

This is not so bad for elliptic curves,

\[ \begin{align*}
E & \quad \mathcal{O}(1) \\
\pi & \quad \text{For any } C \in D^b(E) \\
\tilde{p}_1 \times \tilde{p}_2 & \quad C = \mathcal{O} \oplus ?
\end{align*} \]

\[ \mathcal{O} \oplus \mathcal{O}(2pt) \text{ is not a generator, but a split generator} \]
**Operads** encode "kinds of algebraic structures" e.g. for a k-alg. A, a product \( Y^2 \)

\[
\begin{array}{ccc}
\text{take this, put in input } 1 \\
\text{associativity}
\end{array}
\]

so can think of this as \( Y^3 \)

**Definition:** an operad \( O \) in \( C \) is a collection of objects \( O(n) \) together with composition operators \( \circ_i : O(n_1) \times O(n_2) \rightarrow O(n_1 + n_2 - 1) \) for \( n \leq n_2 \)

(mental picture: think composition of trees)

\( C \) can be anything, sets, top. spaces, k-vect., dg k-vect., etc.

\( \text{e.g.: } Ass(n) = \begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \circ \circ \circ \circ \circ \circ \end{array} \) (linearizes to \( k^{n^2} \))

\( \text{e.g.: } Com(n) = \begin{array}{c} \circ \circ \circ \\ \circ \circ \circ \circ \circ \circ \circ \circ \end{array} \) (linearizes to \( k \))

\( \text{e.g.: For vector space } V, \text{ get } \) \( \text{End}(V)(n) = \text{Hom}(V^\otimes n, V) \)

\( \text{Aside, think of these as boxes} \)

\( \circ : \left( \begin{array}{c} \text{\phantom{.}} \\ \text{\phantom{.}} \end{array} \right) \rightarrow \left( \begin{array}{c} \text{\phantom{.}} \\ \text{\phantom{.}} \end{array} \right) \)

\( \text{an equivalent way to do this is non-labelled composition} \)

\( \circ : O(k) \otimes O(n_1) \otimes \ldots \otimes O(n_n) \rightarrow O(n_1 + \ldots + n_n) \)

**Algebra** over an operad \( O \) is a collection of maps \( O(n) \rightarrow \text{End}(V)(n) \) which is a map of operads

A symmetric operad is one where the objects have an
a symmetric operad is one where the objects have an $S_n$ action (this is how you express something like commutativity)

example: Free operad, $\text{Free}(n) = \frac{1}{2} \sum_{\text{all trees w/ leaves}}$

on one binary generator

Example: Lie operad

linear

Free $^\text{linear}$ operad $\bigwedge^2 Y^2 = -2 Y^1 \bigwedge^3 Y + 3 Y^2 \bigwedge^1 Y + 2 Y^3 = 0$

interesting question:

$\text{dim}_k \text{Lie}(n) = ?$

Kontsevich has an interpretation in terms of the configuration space of points on a sphere

"ideal generated"

Example:

$O(n) = \begin{array}{c} 1 \ldots 2 \ldots n \\ n \end{array}$

note: can topologize the set of operations

"little boxes" operad

Example: $M(n) = \overline{\mathcal{M}}_{g,n}^{\text{synth}} \cong \text{compacification of mod. space of curves, n+1 marked points}$

More modern version of little boxes is little discs:

configuration space of $n$-distinct points in $D_2$

just need to specify centers & radii so that two overlaps

Now let $O$ be an operad in Top

$\Rightarrow$ new operad $\bigwedge H^*(\text{Top} \otimes O)$ of graded v.s.
new operad \( H_k \) \( \otimes \) \( C_k \) operad of \( D_2 \) vs.

What is \( H_k \) (little 2-disks)?

\[
D(1) \cong \bigcirc \quad \quad \quad H_k D(1) = k[0] = \langle \text{id} \rangle \\
D(2) = \bigcirc \quad \cong S^2 \quad \quad \quad H_k D(2) = k[0] \otimes k[2]
\]

Look at line between them, push out to boundary

so algebra over \( H_k D \) in graded vs. \( V \)

\[
H_k(\bigcirc) \longrightarrow \text{Hom}(\bigcirc \otimes V, V)
\]

\[
k \langle \ast \rangle \otimes k \langle [-,-] \rangle
\]

associative, not homogenous, not associative

You can work this out and get

1) \([-,-]\) is a Lie bracket of degree -1

2) \( \cdot \) is an associative, commutative multiplication of degree 0

3) \([x,y,z] = x[y,z] + y[x,z] \)

Called a Gerstenhaber algebra

Observation: For any associative algebra \( A \), \( \text{HH}^*(A) \) is a G-algebra, i.e.

\[
\cdot : \text{HH}^n(A) \otimes \text{HH}^m(A) \longrightarrow \text{HH}^{n+m}(A)
\]

\([-,-] : \text{HH}^n(A) \otimes \text{HH}^m(A) \longrightarrow \text{HH}^{n-1+m}(A)
\]

i.e. \( \text{HH}^*(A) \) is an algebra over \( H_k D_2 \)

More simply:

\[
D(2) = \bigcirc
\]
and \( D_1(n) \) deformation retracts onto \( n^k \) distinct points, so \( H_k D_1 = \text{Assoc.} \).

So we have the more general fact:

**Obs.** For any algebra over \( H_k D_1 \), \( A \), \( HH^k(A) \) is an algebra over \( H_k D_w \).

**Deligne Conjecture:** any algebra over \( C_\ast D_1 \), \( A \), \( C^\ast(A) \) is an algebra over \( C_\ast D_2 \),

proven: Kostelc, Kontsevich, ...

hochschild cochains

**Note:** one can check that an algebra over \( C_\ast D_2 \) is an \( A_\infty \) algebra.

**Illustration:** if \( A = C^\infty(X) \), i.e. smooth affine,

then \( HH^m(A) = \Gamma(\Lambda^m T_X) \), the \( \ast = \wedge \).

\([\cdot, \cdot] : HH' \otimes HH' \to HH' \) is Lie bracket.

---

**Intro. to Hochschild cohomology:**

\[ A \otimes_k k \to C^n(A) = \text{Hom}_k(A^{\otimes n}, A) \]

\[ b : C^k(A) \to C^{k+1}(A) \]

\[ b(f(a_1, \ldots, a_n)) = a_1 f(a_2, \ldots, a_n) - f(a_1a_2, a_3, \ldots, a_n) \]

\[ + \ldots + f(a_1, \ldots, a_{n-1}) a_n \]

can think of this pictorially

\[ \begin{array}{c}
\text{and } Y^* \\
\end{array} \]

and \( b^2 = 0 \), \( C^\ast(A), b \) is a complex, \( HH^k(A) = \frac{\ker(b)}{\text{im}(b)} \)

\[ 0 \to \text{Hom}(k, A) \to \text{Hom}(A, A) \to \text{Hom}(A^{\otimes 2}, A) \]

\[ a \mapsto ax - xa \]

\[ f \mapsto (xy) \mapsto xf(y) - f(xy) + f(x)y \]

so \( 1 \)-cocycles are derivations.
1- coboundaries are inner derivations

If A is commutative, then $HH^1(A) = \text{Der}(A, A) = P(A^* T_A)$

2- cocycles are 1st order deformations

Product/Bracket :

- $C^m(A) \otimes C^n(A) \rightarrow C^{m+n}(A)$
  
  $f \otimes g (x_1, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}) = f(x_1, \ldots, x_m) g(x_{m+1}, \ldots, x_{m+n})$

**Ex:** check that $b(fg) = b(f \cdot g) + b(f) \cdot b(g)$

$\Rightarrow$ descends to $HH^*$

**Ex:** if $bf = bg = 0$,

$f \cdot g - g \cdot f = b \cdot h_{f,g}$

$= b \cdot h_{f,g}$

$\exists$ two canonically defined $h, h'$

$\text{deg}(h_{f,g}) = \text{deg} f + \text{deg} g - 1$

then can define

$[f, g] = h_{f,g} - h_{g,f}$

note: $b \cdot [f, g] = 0$ so this gives a class

$[f, g] \in HH^{*+1}$

Note: While $\cdot$ is associative at the chain level, $[\cdot, \cdot]$ does not satisfy Jacobi at chain level but it does at cohomology level

$\Rightarrow$ give definitions,

$h_{f,g}(x_1, \ldots, x_{m+n-1}) = \sum f(x_1, \ldots, x_k) g(x_{k+1}, \ldots, x_{k+n})/x_{k+m+1}, \ldots, x_{m+n-1})$

$h'_{f,g} = \sum g(...f...)$

$C^*(A), [-, -], \ldots$ forms an L∞-algebra

Batalin-Vilkovisky algebras (BV)

G-algebra: graded vector space

* associative, commutative, degree 0

$[\cdot, \cdot]$ Lie, degree 1

$[xy, z] = x[y, z] + y[x, z]$

BV-alg: graded vector space A
Δ assoc. commutative of deg 0
Δ differential operator of order 2, deg -1

second order means instead of always
in x and y
\[ \Delta(xy) - x\Delta(y) - \Delta(x)y = [x,y] \]
i.e. we can associate a bracket to Δ, and in fact we could have taken this to be a def:
\[ \text{BV-alg is a G-alg where the bracket is given explicitly by a } 2\text{nd order } \Delta: A \rightarrow A \]

Example of a G-algebra:
Let X be a C^n manifold, \( A = \bigoplus \Gamma(\wedge^n T_x) \)
with \( \cdot : -\cdot, [\cdot , -\cdot ] \) lie bracket of vector fields
\[ [\cdot , -\cdot ] \] extended to polyvector fields using derivation property

if X is oriented, A is a BV-algebra
\[ \Gamma(\wedge^i T_x) \text{ is isomorphic to } \Gamma(\Omega^{n-i}_x) \text{ via pairing} \]
perfect of orientation
\[ \wedge^i T_x \otimes \Lambda^{n-i} T_x \rightarrow \Lambda^n T_x \]
\[ \wedge^i T_x \equiv \Lambda^n T_x \otimes (\Lambda^{n-i} T_x)^\vee \]
\[ \equiv \Omega^i_x \otimes \Lambda^{n-i} \Omega_x = \Omega^i_x \]
so \( \Gamma(\wedge^i T_x) \equiv \Gamma(\Omega^{n-i}_x) \)

So get
\[ \Gamma(\wedge^i T_x) \rightarrow \Gamma(\Omega^{n-i}_x) \rightarrow \Gamma(\Omega^{n-i}_x) \]
\[ \Delta \]

claim: Δ is an enhancement of the Lie bracket

Try to write this for \( \mathbb{R}^n \), see what you get

Consider: Framed little discs operad \( \mathcal{F}_2 \)
like \( D_2 \), but every disc has marked point on its boundary

identity, new degree -1.
claim: \( H_*(B_*(f)) = H_*(s^f) \cong k[[s]] \oplus k[[t]] \) \( \Delta: \ast \otimes \ast \to \ast \otimes A \ast \ast \)

So an algebra over \( H_*(B_2) \) is a BV algebra, (obviously something to check here)

Q: we know that for any associative algebra \( A \), \( HH^*(A) \) is a \( G \)-algebra.

For what algebras \( A \) is \( HH^*(A) \) a BV-algebra?

(between we needed a trivialization of \( \omega_x = \Lambda^{top} \Omega^1_x \), i.e. nowhere vanishing volume form)

\( \Lambda^1 \otimes \ast \)

Rings \( \iff \) schemes, \( \omega_x = \text{canonical bundle} \)

then the answer is

A: \( A \) has to be a Frobenius algebra / cyclic \( A_\omega \) algebra

Thm: (Trudler, 1975) : if \( A \) is a Frobenius algebra / cyclic \( A_\omega \) algebra, then \( HH^*(A) \) is a BV-algebra (choice of \( \Delta \) depends on choice of pairing)

Def: \( A \) is a Frobenius algebra if it has a non-degen. symm. pairing \( \langle \cdot, \cdot \rangle : A \otimes A \to A \) s.t. \( \langle xy, z \rangle = \langle x, yz \rangle \)

c.g. \( H^*(X), X \) cplct. mfd.

kG group ring for Finite group

encodes this as

A cyclic \( A_\omega \) algebra is an \( A_\omega \) algebra + pairing \( \langle \cdot, \cdot \rangle \) s.t.

\( c_{k+1} : A \otimes k \to k \)

\( c_{k+1}(x_1, \ldots, x_{k+1}) = \langle \nu_k(x_1, \ldots, x_k), x_{k+1} \rangle \)

is a cyclic tensor, i.e.

\( c_k(x_1, \ldots, x_k) = \pm c_k(x_k, x_1, \ldots) \)

A cyclic \( A_\omega \) algebra with only \( m_2 \) is a Frobenius algebra

VISTA: (Costello)

(1) An open \((d+1)\)-TFT \( \iff \) Calabi-Yau \( A_\omega \) category \( C \)
b) the closed string sector $\leftrightarrow HH_{\ast}(C)$

Glossary:

- $A_{\infty}$ algebra
- assoc. algebra
- homotopy $G$-alg. $\leftarrow C^\ast(A_{\infty})$
- $G$-alg. $\leftrightarrow HH^\ast(A_{\infty})$
- Deligne conj.
- Gerstenhaber
- homotopy BV-alg. $\leftrightarrow C^\ast(CY_{K_{\omega}})$ (Costello)
- BV-alg. $\leftrightarrow HH^\ast(CY_{K_{\omega}})$ (Tralder)

arxiv: Ed Segal '09, First section
Day 4
Thursday, June 11, 2009
9:05 AM

2d open-closed TFT

monoidal Functor from certain category into Vect.

String objects: (oriented) ordered sets of
"closed strings": circles $\mathcal{C}$
"open strings": $A \rightarrow B$

$\Lambda$ set of labels (fixed, i.e. $\exists$ a string category on any set $\Lambda$)
morphisms: (oriented) 2-mfds with corners

free boundary

incoming outgoing

String is monoidal $\otimes := \amalg$ disjoint union
is rigid $\iff$ i.e. has duals (by flipping orientations)
composition = gluing

Claim: every morphism is a composition of the following basic morphisms

$P_c = \quad P_o (A,B,C) = \quad$  
$D_c = \quad I_o(A,B) = \quad$  
$I_c = \quad$  

Defn: Open-closed 2d TFT is a monoidal functor

$\Psi : \text{String} \rightarrow \text{Vect}_k$

Assume we have $\Psi(\emptyset) = V$, and

$\Psi(P_c) = \Psi(D_c) : \Psi(\emptyset \amalg \emptyset) \rightarrow \Psi(\emptyset)$

$\Psi(\emptyset) : V \otimes V \rightarrow V$ this is an associative product because

$\Psi(\emptyset) : V \rightarrow k$ is a trace
\( \Psi(\omega) \cdot v \rightarrow k \) to a trace

\[ \Psi(\mathcal{D}) \] is an inner product

then: this turns \( V \) into

a Frobenius algebra!

Now the open part, for \( A, B \in \Delta \)

\[ \Psi(\Delta) \in \text{Vect}_k \] turns \( \Delta \) into a \( k \)-linear category with \( \text{Hom}_\Delta(A, B) = \Psi(A \to B) \)

composition given by

\[ \begin{array}{ccc}
B & \rightarrow & C \\
\downarrow & & \downarrow \\
A & \rightarrow & \text{Hom}(A, B) \otimes \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)
\end{array} \]

Claim: a "universal" open-closed TFT with a fixed \( k \)-lin. category \( \Delta \) as the open sector is given by

\[ V = \text{HH}(\Delta) \]

Claim: an open-closed 2d TFT is determined by \((\Delta, V)\)

w/ some additional data

now gives \( \text{Hom}(A, B) \otimes \text{Hom}(B, A) \rightarrow \text{Hom}(A, A) \rightarrow k \)

which can also be shown to be a perfect pairing

i.e. we have an identification \( \text{Hom}(A, B) \cong \text{Hom}(B, A)^{\vee} \)

called a Serre functor

name comes from Serre duality, i.e. on a Calabi-Yau \( n \)-fold,

\[ \text{RHom}(A, B) \cong \text{RHom}(B, A[n])^{\vee} \]

A \( k \)-lin. cat. \( \Delta \) with this data is called CY

also have the way the open and closed sectors interact,

which is determined by

\[ A \begin{array}{ccc}
\rightarrow & \Psi(A - A) & \rightarrow \Psi(A) \\
\rightarrow & \text{End}_\Delta(A) & \rightarrow V
\end{array} \]

with compatibility (see below)

\[ \text{acts by conjugation} \]

Example: \( G \) Finite group, \( \Delta = \text{Rep}_k G \)

\[ V = \mathbb{C}[G]^G \] (= class functions on \( G \))

"gauge theory with symmetry group \( G \)"

has something to do w/ principal \( G \)-bundles

in \( V = \mathbb{C}[G]^G \cdot \) multiplication from the group algebra

\[ \iota \text{ as usual} \]

\[ \tau : \mathbb{F} \rightarrow \mathbb{F}(e) \cdot \frac{1}{|G|} \]

so \( \Psi(\mathbb{F}) : \mathbb{F} \rightarrow \mathbb{F} \) is multiplication by \( \frac{1}{|G|} \).
The "whistle" map is $\text{End}_{\text{rep}_G}(A) \rightarrow CG \rightarrow \text{id}_A \rightarrow X_A$

For irreps, otherwise write $A = \bigoplus n_i A_i$

$CG \rightarrow \text{End}_G(A)$, $CG = V \cong \text{End}(\text{id}_{\text{rep}_G})$

under this identification, get

$\text{End}(\text{id}_{\text{rep}_G}) \rightarrow \text{End}_{\Delta}(A)$ \uparrow \cong \text{n}_A

so $V = \mathbb{Z}(\Delta)$ is somehow universal

Complete list of conditions:

$\text{Strng}_c \xrightarrow{-} \text{Strng}_c \xrightarrow{\text{Comm. Frob. alg.}} \text{Vect}$

Want an adjoint $L_{\text{in}}} \rightarrow \text{i}^*$

if $\Delta$ were an open TFT, then $L_{\text{in}}\Delta$ will be the universal open closed TFT with open sector $\Delta$

**Costello:** IF $\Delta$ is an open TFT given by a CY category $\mathcal{E}$ then $\text{i}^* L_{\text{in}}\Delta$ is given by $\cdots$

Note: actually he did a more involved version, just replace

$\text{TFT} \rightarrow \text{TCFT}$ conformal, i.e. morphisms w/ metric up to conformal equivalence

$\text{Vect} \rightarrow \text{dg-Vect}$

$\text{CY cat.} \rightarrow \text{CY A}_\infty$ cat.

$\mathbb{Z}(\mathcal{E}) \rightarrow \text{HH}(\mathcal{E})$

Compatibility between open & closed sectors:

$A \rightarrow 0$

$\text{Hom}(A, A) \xrightarrow{i^*} V$, $\forall A$

$V \xrightarrow{\text{c}^*} \text{Hom}(A, A)$

a) $i^*$ is a ring map whose image is in $\mathbb{Z}(\text{Hom}(A,A))$
a) \( \psi^0 \) is a ring map whose image is in \( \mathbb{Z}(\text{Hom}(A,A)) \)
b) the Cardy condition is satisfied:

\[
\begin{align*}
\text{Hom}(A,A) \otimes \text{Hom}(B,B) &\rightarrow V \\
\downarrow \cong \psi^0 \otimes \psi^0 &\rightarrow V \\
V \otimes V &\rightarrow \text{in } V
\end{align*}
\]

This must give the same map, because it is a diffeomorphic shape.

\[\text{break}\]

Assume \( \Lambda = \mathbb{Z} \oplus \mathbb{Z} \) i.e. "one brane"

\( \Rightarrow \text{Hom}(\ast, \ast) = \Lambda \) is now just a ring

\( \mathcal{V}(\ast) \)

This is a (not ncy commutative) Frobenius algebra, but \( \langle , \rangle \) is symmetric

\[\text{Claim: the most general octFT will have.}\]

\[V = \text{span} \left\{ \text{all oc diagrams with open inputs labelled by elements of } \Lambda \text{ and one closed output} \right\}\]

/all ncy relations (i.e. diffeomorphisms)

Note: inputs labelled because elements + diagram gives an element of \( V \)

\[\Lambda \otimes \Lambda \rightarrow V \]

\[\Lambda \otimes \Lambda \rightarrow \text{some vector}\]

Now Cardy condition guarantees that any such diagram is of the form open

\[\text{so we can reduce to the square}\]

Penn Conference Page 4
\[ V = \text{span} \left\{ \text{all open diagrams } \right\} + \text{one pennywhistle } / \text{relations} \]

\[ \text{e.g. } \]

\[ \begin{array}{c}
\text{cardy} = \\
\text{huge mess}
\end{array} \]

Now we think of elements of \( V \) as (diagram & inputs in \( \Delta \)), and the relations between them are

\[ \text{Diagram}_1(\Delta, \Delta') = \text{Diagram}_2(\Delta, \Delta') \]

if they have the same output in any octTFT.

In particular if we have some complicated diagram

\[ \Delta \rightarrow \Delta' \]

So we can take

\[ V = \text{span} \left\{ \Delta \rightarrow \Delta' \right\} / \text{relations} \]

claim that multiplication on \( V \) can be given by multiplication of the labels \( \Delta \in \Delta \), and that the only relations are \( [\Delta, \Delta] < \Delta \).

So far we have argued that for the universal octTFT, \( \Delta/\Delta, \Delta] \rightarrow V \).

Now one argues that on \( \Delta/\Delta, \Delta] \) there is an open-closed TFT structure, and it follows that it must be the universal one.

What is Frobenius pairing on \( \Delta/\Delta, \Delta] \)? It is ultimately given by \( \langle a, b \rangle = \text{Tr}(x \rightarrow axb) \) as a linear map \( \Delta \rightarrow \Delta \), i.e.

\[ \text{doesn't depend on the trace of } \Delta! \]

take \( a, b \in V \), cardy
now we have the

**Thm:** let $\Lambda$ be a symm Frobenius algebra and assume
$\langle a, b \rangle = Tr(x \mapsto axb)$ is a non-degen. pairing on
$\Lambda/[\Lambda, \Lambda] = HH_0(\Lambda)$ then $(\Lambda, \Lambda/[\Lambda, \Lambda])$ is the
universal octTFT with open sector $\Lambda$

Note on Hochschild homology $HH_0(\Lambda) = \Lambda/[\Lambda, \Lambda]$
Calabi-Yau (Frobenius) condition on $\Lambda$ lets you show that
$HH_*(\Lambda) \cong \mathbb{Z}(\Lambda)$

in general for CY categories $HH_*(X) \cong HH^{*+i}(X)$
actually CY condition gives you an isomorphishm at the level of chain complexes

Reference: arxiv: 0412149 section 4.5 (Costello)

$CY$ condition $\Longleftrightarrow$ identification between $C_*(\Lambda), C^*(\Lambda)$
Good:
1) $HH^*(\Lambda)$ is a ring but no pairing because $HH_*(\Lambda)$ has pairing but no multiplication
2) $HH_*(\Lambda)$ has pairing but no multiplication for any $\Lambda$
3) $A= "complex manifold" X, HH^*(X)$ is a ring
   $HH_*(X)$ has pairing if $X$ is compact

$CY$ condition $\Longleftrightarrow$ orientability for $H^*$ for $H_*$

$HKR$ $HH_*(X) \cong \bigoplus_{p,m} H^p(X, \Omega^m)$

$HH_n(X)$
Now how does CY condition give a pairing if A is Frobenius (=CY)

\[ C_x(A) = A^{\otimes x+1} \]

\[ \ldots \rightarrow A^{\otimes 3} \rightarrow A^{\otimes 2} \rightarrow A \rightarrow 0 \]

- \( \text{able} \rightarrow \text{able} - \text{albc} + \text{calb} \)
- \( \text{alb} \rightarrow \text{ab} - \text{ba} \)

Dualizing this,

\[ 0 \rightarrow A^* \rightarrow (A^{\otimes 2})^* \rightarrow \ldots \]

Then use the pairing to get

\[ 0 \rightarrow \text{Hom}(k, A^*) \rightarrow \text{Hom}(A, A^*) \rightarrow \text{Hom}(A^{\otimes 2}, A^*) \rightarrow \ldots \]

and compatibility guarantees you get \( C^*(A) \)
Day 5  
Friday, June 12, 2009  
9:28 AM

arrived a little late...

1) From TFT to TCFT (Costello’s paper)  
2) GW theory for \(A_\infty\) algebras (Costello’s 05)

Thm: (Strebel & Penner) \(H_k(M_{gn}) \cong H_k\) (Ribbon graph complex)  
for \(n \geq 1\)

\[
\begin{align*}
M_{0,3} &= pt \\
M_{0,4} &= S^3 \setminus 3 pts \\
M_{0,5} &= (\mathbb{RP}^1 \times \mathbb{RP}^1) \setminus 7 \text{ lines}
\end{align*}
\]

Ribbon graph: graph \(P\) + cyclic ordering of half-edges  
(assume valency of each \(\geq 3\))

\[
\begin{tikzpicture}
\draw (0,0) circle (0.5cm);
\draw (1,0) circle (0.5cm);
\draw (2,0) circle (0.5cm);
\draw (3,0) circle (0.5cm);
\draw (0,0.5) -- (1,0);
\draw (1,0) -- (2,0);
\draw (2,0) -- (3,0);
\end{tikzpicture}
\]

note: a planar diagram gives cyclic ordering

A ribbon graph \(\Leftrightarrow\) two perm. of \(S_3\) set of half-edges \(Z = \sum_{e \in \sigma} \sigma e\)

\(\sigma\) encodes cyclic data at vertices,  
\(\tau\) encodes how edges are attached

where:  
vertices = cycles of \(\sigma\)  
edges = cycles of \(\tau\)  
factors = cycles of \(\sigma \tau\)

Genus \((P) = g \iff 2 - 2g = v - e + f\)

Thm: every ribbon graph gives a top. surface of  
genus \(g\) w/ \(f\) boundary components

\[
\begin{tikzpicture}
\draw (0,0) circle (0.5cm);
\draw (1,0) circle (0.5cm);
\draw (2,0) circle (0.5cm);
\draw (3,0) circle (0.5cm);
\draw (0,0.5) -- (1,0);
\draw (1,0) -- (2,0);
\draw (2,0) -- (3,0);
\end{tikzpicture}
\]

degree \((P) = \sum_{\text{vertices}} Val(r) - 3\)

so define \(Gr = \mathbb{C}(\text{ribbon})\) graded by degree, \(d\), and have a differential

\[d: Gr_n \rightarrow Gr_{n-1}\]

\[d(P) = \sum_{r} \pm P \text{ expanded at } r\]
\[ \text{v-vertices(}P\text{)} \rightarrow \text{all ways to expand } P \text{ at } v \]

i.e. \[ \times \rightarrow \times \times \times , \text{ this always decreases the degree by 1} \]

can also express \[ d(\pi) = \sum_{\pi' \text{ and } c \in \pi'} \pi' \text{ with } \pi'^c = \pi \]

\[ \text{signs chosen so that } d^2 = 0 \]

Now \((Gr, d)\) is graph complex

e.g. \(g=0, n=3\)

\[ v-e+3 = 2 \Rightarrow v-e = -1 \]

so \(v \leq 2, 3v \leq 2e, 3v-3e = -3\)

\[ C \rightarrow C^2 \]

\[ 8 \rightarrow \emptyset \]

\[ 1 \rightarrow (\underline{1}) \]

\[ H_0(Gr) = C \]

\[ H_1(Gr) = 0 \]

\[ \Rightarrow H_*(Gr) \cong H_*(M_{g,3}) = H_*(pt) \]

Note, this won't work for higher \(g, n\) because under this notion of isomorphism, the "marked points" can be permuted, whereas in \(M_{g, n}\) they must be fixed. So we should add the extra data of labelling the facets of \(P\) and only identify graphs with corresponding labelings.

There is a stronger statement involved:

**Def:** A metric ribbon graph is a ribbon graph \(P\) and an assignment of positive lengths \(c \in \mathbb{R}^+\) to its edges.

\[ \Rightarrow \text{can construct a moduli space of metric r.g.'s} \]

\[ \text{note } \begin{array}{c} a \\mapsto \\emptyset \text{ as } c \to 0 \end{array} \text{ in } M_{g, n} \]

with labels, \(M_{g, 3} \cong \mathbb{R}^3\), given by

on facets diam : \(M_{g, 3} \rightarrow \mathbb{R}^3\)

\[ \text{take facet } i \text{ to its diameter (sum of lengths of edges around)} \]
Thm: (different proofs by Mumford, Strebel, Penner) \( M_{g,n} \cong M_{g,n} \times \mathbb{R}^+ \)

(homeomorphic)

(note: used by Kontsevich to prove the Witten conjecture for the GW theory of a point)

the key this fits in

\[ \text{TCFT} \leftrightarrow C^*_k(M_{g,n}) \leftrightarrow \text{ribbon graph} \leftrightarrow A_{\infty} \text{-categories} \]

\[ \text{start almost a definition, hard part conceptual, but not so much work} \]

Open TFT := \( \psi: \{ \text{top surfaces w/ open boundary} \} \rightarrow \text{Vect} \)

Open CFT := \( \psi: \{ \text{riemann surf. w/ open boundary} \} \rightarrow \text{Vect} \)

Open TCFT := \( \psi: C^*_k(M) \rightarrow \text{dg-Vect} \)

(moduli space of Riem. Surf. w/ open boundary)

every \( \sigma \rightarrow \text{dg-Vect} \otimes \text{Hom}^*(A, B) \)

A chains in \( C^*_k(M, (A \oplus B) \otimes \text{Hom}^*(A, A) \otimes \text{Hom}^*(B, B) \rightarrow V^* \)

---

Now on the other side:

How to take a ribbon graph \( \Gamma \) and a cyclic \( A_{\infty} \) algebra \( A \) and produce a number (called morsification)

cut so that every layer has only one vertex, cap, or cap

Form a complex based on the number of intersections with each cut
Another way to get this: pick basis $e_1, \ldots, e_n$ of $A$.

$$
ev(G, A) = \sum_{\text{all ways to label half edges of } G} \sum_{\text{vertices with adjacent half edges labeled } e_i, e_j} C_{ik}(e_i, e_j)$$

Thm: $\ev(dG, A) = 0$

$$d \left( \begin{array}{c} 1 \\ -1 \\ 2 \\ 3 \\ 4 \end{array} \right) = \left( \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \right)$$

$\text{Asscr } \Rightarrow \ev(d(x), A) = 0$

thus $\ev(-, A)$ gives a cocycle on (ribbon graph $C_\times$)

$C_\times(M_{\Sigma n}/\Sigma_n)$

$\Rightarrow$ a cyclic $A_\infty$-algebra

gives rise to a class in

$H^*(M_{\Sigma n}/\Sigma_n)$

Spruce this up a bit:

Ribbon graph $G \leftrightarrow$ (chain of)
surfaces with $n$-punctures

Note we want

$$C_\times(\text{closed } M_n) \otimes V^\otimes n \rightarrow C$$

what should $V$ be?

$$C_\times(\text{closed } M_n) \otimes \text{ of } C_\times(A) \rightarrow n \times \text{ of } C_\times(A)$$

(Kontsevich-Solbelman)

Note: Costello argues that for the moduli of complex structures on an underlying top space

$M$ up to homotopy

Costello: "a dual ribbon graph construction of the moduli space of curves"
More precisely, the correspondence goes
Open TCFT $\leftrightarrow C_*(\text{open } \mathcal{M}_{g,n}) \sim C_{\text{moduli of}}$
$\text{top surfaces of}$
$\text{cplx str. on}$
$\text{plane}$
$\text{discs}$
$\text{top surfaces}$
$\text{of}$
$\text{discs}$
$\text{of}$

$A_{\infty}$ categories $\leftrightarrow C_*(\text{ribbon graphs with outside})$

$\text{Hope: (best of all possible worlds)}$

Mirror symmetry:
$B(\mathbf{X}) \sim A(\mathbf{Y})$ def. in $\mathcal{F}_X$
$\text{category}$
$\text{called Fukaya}$
$\text{category}$
$A_{\infty}$-categories

$\text{Hom}^*(E,F) = \mathcal{U}^{\text{res}}(E^\vee \otimes F) \to \mathcal{U}^\partial(E^\vee \otimes F) \to \mathcal{U}^{\partial^2}(E^\vee \otimes F) \to \cdots$

$\to \mathcal{H}^*(E^\vee \otimes F)$

So $B(\mathbf{X})$ is complexes of holomorphic vector bundles
on $\mathbf{X}$, with
$\text{Hom}^*_B(E,F) = \text{Ext}^*_X(E,F)$

originally defined to explain some numerical invariants
$GW(\mathbf{X})$ is a sequence of integers indexed by $g$ degree $d$
$\text{original conjecture was } GW(\mathbf{X}) = \text{Hodge invariants } (\mathbf{X})$

Want to have $GW(A_{\infty}\text{-category})$ so that
$GW(\text{Fuk}(\mathbf{X})) = GW(\mathbf{X})$
$GW(\text{DG}(\mathbf{X})) = \text{Hodge invariants } (\mathbf{X})$

what are the usual $GW$ invariants? stable
$\mathcal{M}_{g,n}(\mathbf{X})$ is moduli space of maps from
a complex curve $C \to \mathbf{X}$
i.e. Riem. Surf.

not compact, so
Kontsevich constructs
$\overline{\mathcal{M}}_{g,n}(\mathbf{X})$ compactification, has a "perfect obstruction theory"$
$
When $\mathbf{X}$ is CY 3-fold, exp. dim $\overline{\mathcal{M}}_{g,n}(\mathbf{X}) = 0$
so $GW_{g}(X) = \int_{M_{g}}^{vir} \sim$ just a count of points

ev(\mathfrak{P}, \mathfrak{A}) \leq C

How to extend ribbon graph picture from $M_{g,n}$ to $\overline{M}_{g,n}$

gave a different idea

\text{Sen-Zwiebach} (\sim 92-93)

\text{chain in $\overline{M}_{g,n}$}

give a chain $\nu_{g,n} \in \mathcal{C}_{even}(\overline{M}_{g,n})$ s.t. if there were a theory on $\overline{M}_{g,n}$ then $\nu_{g,n}$ called "string vertices"

some ideas

$[\overline{M}_{g,n}]$ are solutions to QME on $\mathcal{C}_{even}(\overline{M}_{g,n})$

QME: if $A$ is a BV algebra, $d, \cdot, \Delta; \Delta^{2} = 0$, $d^{2} = 0$

QME: $(d + \Delta) \exp(S) = 0$

$dS + \Delta S + \frac{1}{2} \Delta^{2}S, S^{2} = 0$

solutions exist, any given initial conditions, any two solutions of QME are homotopic

(think of $A$ as something like polyvectorfield algebra)

So to get "string vertices," solve QME on $\overline{M}_{g,n}$ (up to homotopy they would map to the desired virtual classes in $M_{g,n}$)

giving a BV alg. structure on $\mathcal{C}_{even}(\overline{M}_{g,n})$

using ribbon graphs:

\[ x \cdot y = \sum_{\text{disjoint union}} x \in x, y \in y \]

$\Delta(x) = \sum_{x \in x} x \in x$, pick any two marked points

and twist-sew them:

$\Delta: C_{k}(\overline{M}_{g,n}) \rightarrow C_{k+2}(\overline{M}_{g,n-2})$

$+1$ because we get a whole family of ways to sew.
\[ \Delta(x) - \Delta(x)y - x\Delta(y) \]

\[ \exists x, y \]

How do we get \( \partial \overline{M}_{g,n} \)?

\[ \xrightarrow{\exists} x \]

\[ \Delta : \overline{M}_{g,n} \rightarrow \partial \overline{M}_{g,n} \]

This is the whole boundary of \( \overline{M}_{g,n} \)

\[ 2\overline{M}_{g,n} = \frac{1}{2} \sum \overline{M}_{g,n} \rightarrow \overline{M}_{g,n} + \Delta \overline{M}_{g,n} \]

so \( \sum_{g,n} [\overline{M}_{g,n}]^{\text{vir}} \) satisfies QME.