Gauge Thy and Wild Ramification

hep-th/ 051109, 0631
G-bundles with wild singularities

- Gukov and Witten 0612073
- Kapustin and Witten 0604151
  Electric-Magnetic Duality and the Geometric Langland's program

⇒ G-bundle with connections on curves
with tame singularities
regular logarithmic

Moduli spaces of connections

\[ G = \text{GL}(n, \mathbb{C}) \text{ or } \text{SL}(n, \mathbb{C}) \]
\[ E \text{ a smooth complex projective curve of genus } g \]
\[ P_1, \ldots, P_s \in \mathbb{P} \text{ if } P_i \neq P_j \]

\[ E \to \mathbb{C} \text{ a holomorphic vector bundle of rank } n \text{ at most} \]

Def: A connection on \( E \) with logarithmic singularities is
a bundle map
\[ \mathcal{D} : E \to E \otimes T^*_{\mathbb{C}}(P_1 + \ldots + P_s) \]
satisfying Leibnitz rule, that is

For any \( f \in \Omega \), \( \sigma \in \mathcal{E} \)

local section \( \nabla(f\sigma) = df \otimes \sigma + f\nabla(\sigma) \)

\( T^\mathscr{E}_{\mathcal{E}}(\mathcal{P}_1 + \ldots + \mathcal{P}_s) \) a sheaf of meromorphic 1-forms on \( \mathcal{C} \) which have at most order -1 poles at \( \mathcal{P}_1, \ldots, \mathcal{P}_s \)

\( z_i \) local coordinate at \( \mathcal{P}_i \) s.t. \( z_i(\mathcal{P}_i) = 0 \)

\[ \nabla = \frac{dz + A_i(z)}{z_i} \]

\( A_i(z_i) \) holomorphic n\times n matrices near \( z_i = 0 \)

\[ A_i(z_i) = A_{-1} + z_i A_0 + z A \]

\[ \nabla = \frac{dz + \left( \frac{A_i(z)}{z_i} + A_0 + z_i A_1 + \ldots \right)}{z_i} \]

\( A_i \in M(n, \mathbb{C}) \) n\times n matrices
Def \[ \text{Res}_{p_i} \nabla = A_{-1} \]

Ex \[ \text{Res}_{p_i} \nabla \text{ is independent of the choice of the local coordinates } z \]

\[ \Delta = \{ t \in \mathbb{C} \mid |z| < 1 \} \]
\[ w = w(z) \text{ another coordinate} \]
\[ w(0) = 0 \quad w = q_1 z + q_2 z^2 + \ldots \]

\[ \nabla = d + \frac{dz}{z} A(z) = d + \frac{dw}{w} B(w) \]

\[ A(0) = B(0) \text{ \quad just \ try} \]
\[ \frac{dn}{n} = d \log z \]

We see that for log connection

\[ \nabla : E \rightarrow E \otimes T^*_C (p_1 + \ldots + p_s) \]

\[ \text{Res}_{p_i} (\nabla) \quad i = 1, \ldots, s \]

are important invariants

\[ E |_{\Delta_i} \xrightarrow{\sim} \Delta_i \times \mathbb{C}^n \]
\[ \nabla |_{\Delta_i} \sim \Delta_i \]

\[ \text{Rep}_{p_i} (\nabla) \in \text{End}(\mathbb{C}) \]
Eigenvalues of $\text{Rep}_{p_i}(\nabla)$ are independent of choice of trivialization of $E$ at $p_i$.

Say $\lambda_{1,i}, \ldots, \lambda_{n,i}$ are eigenvalues of $\text{Rep}_{p_i}(\nabla)$ we pick order of eigenvalues.

Most natural invariant for $\nabla$ arising from $\text{Res}$

Characteristic Polynomial of $\text{Res}_{p_i}(\nabla)$

$$
\prod_{j=1}^{n} (x - \lambda_{j,i}) = \chi(\text{Rep}_{p_i} \nabla)(x)
$$

Def. Let $\nabla : E \to E \otimes T^*_C(p_1 + \ldots + p_s)$ be a log connection on $C$.

(1) $\nabla$ is called resonant at $p_i$ if two eigenvalues of $\text{Res}_{p_i}(\nabla)$ coincide.

(2) $\nabla$ is called

$$
\nabla : E \to E \otimes T^*_C(p_1 + \ldots + p_s)
$$

$$
\deg E = \deg (\Lambda^n E) = c_1(E) \in H^2(C, \mathbb{Z}) \approx \mathbb{Z}
$$

Lemma (Fuchs relation)

$$
\sum_{i=1}^{s} \left( \sum_{j=1}^{n} -\lambda_{j,i} \right) = -\deg E \in \mathbb{Z}
$$
For simplicity, now we assume that $n=2$

$$\text{rank } E = 2$$
$$\deg E = 0$$
$$G = SL(2, \mathbb{C})$$

$$\text{Resp}_i(D) \in \mathfrak{so}(2, \mathbb{C})$$

$$\gamma_{1,i}, \gamma_{2,i}, \gamma_{1,i} = -\gamma_{2,i} = \gamma_i$$

**Rank 2 $5\ell_2$ case**

$D$ is called reducible iff

$$\sum_{i=1}^{s} \varepsilon_i \gamma_i \in \mathfrak{g}$$

$$\varepsilon_i = \pm 1$$

$$(E, D) > (L, D)$$

$$\nabla / L = \nabla_{L}$$

$$D(L) \subset L \otimes T_{c}^\ast (p_1 + \ldots + p_s)$$

$$\sum_{i=1}^{s} \text{Resp}_i(D_L) = - \deg L \in \mathbb{Z}$$

$$\text{Resp}_i(D_L) \lambda_i \rightarrow L \pi_i$$

$$\mathcal{E}_{\pi} \leadsto \mathcal{E}_{\pi}$$
E complex nb on C
\[ c_1(E) = c_1(\Lambda^n E) \in H^2(C, \mathbb{Z}) = \mathbb{Z} \]

\( L_i = \Lambda^n E \quad \{ g_{ij} \} \quad C = U_d \quad U_d \)

\( \{ g_{ij} \} \in H^1(C, \mathcal{O}_C^\times) \)

\( d(\log g_{ij}) = \omega_d - \omega_b \quad \text{mero. one-form} \)

\[ \frac{1}{2\pi i} \log g_{ij} = \frac{1}{2\pi i} (\omega_d - \omega_b) \]

\( \mu_b - a_d \quad \mu_d \quad \text{Coo 1-form on } U_d \)

\( \Phi = \frac{1}{2\pi i} \omega_d + a_d \quad \text{Coo 1-form on } U_d \quad \text{global 1-form on } C \)

\[ d\bar{\Phi} = da_d \quad \text{2-form} \quad H^2_{DR}(C, \mathbb{Z}) = H^2(C, \mathbb{Z}) \]

\[ = c_1(\Lambda^n E) \quad \text{Weil - De Rham correspondence} \]
\[
\int_{\gamma} \frac{1}{z} = \int_{C \setminus \{A_1, \ldots, A_5\}} \frac{1}{z} d\Phi
\]

\[
\sum_{i=1}^{s} \text{Res}_i (\Lambda^\alpha \mathcal{E}) = \sum_{d=1}^{s} \gamma_{j_d, i}
\]
- Moduli spaces of connections with logarithmic singularities on curves - $M$

- Moduli spaces of representations of $\pi_1(C \setminus \{p_1, \ldots, p_s\}) \to \text{R}$

  $R_H: \omega \to \text{R}$

  local system $\iff \rho: \pi_1(C \setminus (p_1, \ldots, p_s))$

  $(E, \nabla) \to (\ker(\nabla^*|_{C \setminus (p_1, \ldots, p_s)}), \text{GL}(n, \text{C})$

  $\nabla: E \to E \otimes T^*_c(p_1, \ldots, p_s)$  flat connection

  $R(\nabla):$ curvature of $\nabla$

  $\nabla_\omega \sigma = 0$

  $\nabla_\sigma = 0$

**General Ideas**

The Riemann-Hilbert correspondence $R_H$ is an analytic isomorphism

- natural space
- complex str.
- complex algebraic variety

$R_H: M \to \text{R}$

There are several problems to have good moduli spaces

To avoid difficulties, one has to use Mumford Geometric Invariant Theory
\[ \text{algebraic variety} \]

\[ \mathbb{C} \backslash G_C \leftarrow \text{algebraic group} \]

\[ \mathbb{R}_+ : \mathbb{M} \rightarrow \mathbb{R} \]

\[ \uparrow \quad \text{smooth} \quad \uparrow \quad \ast \text{singularities} \]

Need to be careful

\[ C : \text{a smooth algebraic curve of genus } g \]

\[ E : \text{holomorphic vector bundle of rank } r \]

\[ T^*_C \rightarrow \Omega^1_C \text{ sheaf of hol } 1\text{-forms} \]

\[ t_1, \ldots, t_n \in C \quad \text{iff} \quad t_i \neq t_j \]

\[ \mathfrak{b} = \{ t_1, \ldots, t_n \} \quad (C, \mathfrak{b}) \text{ genus } g \text{ curve w/ } n \text{ points} \]

\[ \mathfrak{b} \rightarrow D(\mathfrak{b}) = t_1 + \ldots + t_n ; \text{ divisor of } \mathfrak{b} \text{ on } C \]

\[ \nabla : E \rightarrow E \otimes \Omega^1_C (D(\mathfrak{b})) \]

\[ \text{res}_{t_i}(\nabla) = \text{has eigenvalues} \quad \{ \lambda^{(i)}_0, \lambda^{(i)}_1, \ldots, \lambda^{(i)}_{n-1} \} \]
$\mathcal{W} = \{ \nu^{(i)}_j \} \quad \forall i \leq n$

the set of local exponents of $\nabla$

**Fuchs relation**

\[
\sum_{i=1}^{n} \left( \sum_{j=0}^{r-1} \nu^{(i)}_j \right) = -\deg E \quad e \in \mathcal{W}
\]

\[
\mathcal{W}^n(d) = \left\{ \mathcal{W} = (\nu^{(i)}_j) \mid \text{Fuchs} \right\}
\]

\[\simeq \mathbb{C}^{nr-1}\]

*(GIT)* Parabolic str on $(E, \nabla)$

Fix the data $(C, \pi_C)$

$(E, \nabla, \{ l^{(i)}_+ \}) \quad 1 \leq i \leq n$

Parabolic connection with local exponents $\mathcal{W}$

$D : E \to E \otimes \Omega^1_C (D(x))$

$\deg E = d$

$\text{rank } E = r$

(2) $l^{(i)} = E_0 > l^{(i)}_1 > l^{(i)}_2 > \ldots > l^{(i)}_r > l^{(i)}_r$ = 0

for each $i$, $1 \leq i \leq n$

s.t. $\dim \left( l^{(i)}_j / l^{(i)}_{j+1} \right) = 1$

$\left( \text{Res}_{t_i} (D) - \nu^{(i)}_j \text{Id} \right) l^{(i)}_j \subset l^{(i)}_{j+1}$
\[ \text{Stability condition for } (E, D, \{\ell_i\}) \]

Weights \( \alpha = \{\lambda(i)\} \) \( 1 \leq i \leq r \)

\( 0 < \lambda_1^{(i)} < \lambda_2^{(i)} < \ldots < \lambda_r^{(i)} < 1 \)

\( (i, j) \neq (i', j') \quad \lambda_i^{(j)} \neq \lambda_i^{(j')} \)

**Def** \( (E, D, \{\ell_i\}) \) parabolic connection is stable

\[ \forall (\xi, 0) \in \mathbb{F}, D\xi \neq (E, D) \]

Sub connection
\[
\deg F + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{\alpha (1) \log |V_i|}{\log |F_i|} \right)^j \]
\[
\text{rank } F
\]
\[
< \deg E + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{\alpha (1) \log |V_i|}{\log |E_i|} \right)^j
\]
\[
\text{length}
\]
\[
\text{len (F)}_{j}^{(i)} = \dim \left( \frac{F_{i; \mathcal{L}^{(i)}}}{\mathcal{P}_{i; \mathcal{L}^{(i)}}} \right) = \{ 0 \}
\]
\[
\text{w/o } \mathcal{F} \rightarrow \text{regular slope stability}
\]
\[
\text{parabolic str}
\]
\[
\text{Max}_{(C, \mathcal{F})} \left( \mathcal{D}, \mathcal{R}, n, d \right) = \left\{ \left( E, \mathcal{D}, \mathcal{S}^{(i)} \right) \mid \begin{array}{l}
\text{an } \mathcal{L}-\text{stable } \mathcal{D}-\text{parabolic connection on } C \\
\text{of degree } d \text{ and rank } r
\end{array} \right\}
\]
\[
\text{Thm} \quad (\text{Inaba, Inaba, Ina - Siu-To})
\]
\[
\text{For fixed } (C, \mathcal{F}) \text{ w } \mathcal{F} \subset \text{generic}
\]
\[
\text{Max}_{(C, \mathcal{F})} \left( \mathcal{D}, \mathcal{R}, n, d \right) \text{ becomes a non-singular}
\]
\[
\text{algebraic variety with holomorphic symplectic structure}
\]
\[
\text{of dim}
\]
\[
= 2r^2(g-1) + nr(r-1) + 2
\]
\[
\mathcal{D} \text{ not resonant}
\]
\[
\left( E, \mathcal{D} \right) \left( \not \in \left( E, \mathcal{D} \right) \right)
\]
Abelian Hopf

\[ M \overset{\text{biregular}}{\longrightarrow} M_{#} \]

\[ M_{#} \overset{\text{biregular}}{\longrightarrow} M_{#'} \]
\[ T_{g,n} = \{ (C, \pi) \} \]

\[ M^d(r, d, n) \]
\[ \downarrow \quad \text{smooth morphism} \]
\[ \overline{T_{g,n}} \times N_{C}^d(d) \cong (C, \pi), \mathcal{V} \]

s.t.
\[ \pi^{-1}((C, \pi), \mathcal{V}) \cong M_{(C, \pi)}^d \mathcal{V}(Z, r, \psi, d) \]

\[ g = 0 \quad C \cong \mathbb{P}^1 \]

\[ n = 4 \]

\[ (\Lambda^2 E, \Lambda^2 \nabla) \overset{\text{fix}}{\cong} (\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}) \]
\[ \text{rank } E = 2 \]
\[ \Lambda^2 E = \mathcal{O}_{\mathbb{P}^1}(-1) \text{ degree } -1 \]

\[
\mathcal{M}_{(1-p^1, \ell)}(\mathbb{P}^1, 2, 4, -1) = \mathcal{M}^\infty(\mathcal{V})
\]

\[
\mathcal{V} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 \end{pmatrix}
\]

\[ q : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}'_{\mathbb{P}^1} \]

\[ \mathcal{D} : \mathcal{O}_{\mathbb{P}^1}(-\ell_i) \]

\[ \nabla : E \to E \otimes \Omega'_{\mathbb{P}^1}(t_1 + t_2 + t_3 + t_4) \]

\[ \mathcal{D} = q + \sum_{i=1}^{4} \frac{A_i(q)}{(2-t_i)} \mathcal{D}'(q) = \begin{pmatrix} a_{11}(q) & a_{12}(q) \\ a_{21}(q) & a_{22}(q) \end{pmatrix} \]

\[ \text{GL}(2, \mathbb{C}) \]

\[ \mathcal{M}^\infty(\mathcal{W}) \text{ 2 and smooth variety} \]

\[ \mathcal{M}^\infty = \mathcal{O} \mathcal{M}^\infty(\gamma) \]

\[ \pi \]

\[ \mathcal{W} \times \mathcal{N} \sim (\pi, \mathcal{D}) \]

\[ (t_1, \ldots, t_4) \quad (\gamma_1, \ldots, \gamma_4) \]
\[ V = (v_1, v_2, v_3, v_4) \text{ generic resonant} \]

\[ 2v_i \neq V : \text{ resonant} \]

\[ \Rightarrow \exists i : 2v_i \in V \]

\[ t = (t_1, t_2, t_3, t_4) \in \mathbb{P}^4 \]

\[ C_\infty \]

\[ C_\infty^2 = -2 \]

The 2nd Hirzebruch surface of degree 2,  

\[ F_2 = \text{Proj}(\Omega \oplus \Omega(2)) \]

Blow-up at 8 points to obtain \( S' \) smooth proper surface

\[ M_{\text{ht}}^1 (V) = \mathbb{P}^1 \left( C_\infty + D_1 + D_2 + D_3 + D_4 \right) \]

Delete proper transform

\[ D_i^2 = -2 \]

\[ D_i = \text{first self-intersection} \]

after blow-up \[ -2 \]
Dual graph

\[ D_4 \quad C_7 \quad D_3 \]

\[ D_1 \quad D_2 \]

Affine Dynkin diagram \( D_4^{(1)} \)

\( K_g \) canonical divisor \( A^2 \Omega^1_S = \Omega^2_S \)

\( K_S \sim -2C_\infty -D_1 -D_2 -D_3 -D_4 \)

\( \Rightarrow \) \( \Omega_S \) : rational 2-form on \( S' \)

whose poles are \( 2C_\infty + D_1 + D_2 + D_3 + D_4 \)
(with multiplicity)

\( \Omega_S = S \setminus D_i \) : non-degenerate holomorphic on \( S \setminus D_i \cong M(\alpha)(\gamma) \)
$v_1 = v_1^*$

affine variety cannot contain compact curve besides point -2 curve

Characteristic variety
\[ C \] a non-singular projective curve of genus \( g \geq 0 \)
\[ \mathcal{H} = \{ t_1, \ldots, t_n \} \] a set of \( n \)-distinct points on \( C \).
For \( \mathcal{H} \), we define a divisor

\[ \text{Moduli space of stable parabolic connections} \]

\( E \) an algebraic vector bundle on \( C \) of
rank \( r \)
degree \( d \)

A logarithmic connection on \( E \) is a morphism of
sheaves

\( \nabla: E \to E \otimes \Omega_C^1(D(\mathcal{H})) \)
which satisfies Leibnitz rule: For any local section
\( a \in \mathcal{O}_C \), \( \sigma \in E \)

\[ \nabla(a \sigma) = \sigma \otimes da + a \nabla(\sigma) \]

\( C \)

\[ \nabla = d + \sum \frac{A_i}{n - b_i} \quad \text{locally} \]
\[(E, \nabla) \rightarrow \forall i, \ 1 \leq i \leq n\]
\[\text{res } t_i: (\nabla) \in \text{End}(E|t_i)\]

Eigenvectors \(\{\nu_0^{(i)}, \nu_1^{(i)}, \ldots, \nu_{r-1}^{(i)}\}\)

\[\nabla = \{\nu_j^{(i)}\} \mid 0 \leq j \leq r\]

Set of local exponents of \(\nabla\)

\[\text{Fuchs relation } \sum_{i=1}^{n} \sum_{j=0}^{r-1} \nu_j^{(i)} = -\deg E = -d \in \mathbb{Z}\]

\[\text{Tr}(\text{res } t_i(\nabla))\]

\[\mathbb{C}^{nr} \ni \nabla = \{\nu_j^{(i)}\}\]

Space of local exponents of degree \(d\).

\[\mathbb{C}^{nr-1} = \mathcal{N}_{r}^{n}(d) = \{\nabla \mid d + \sum_{i=1}^{n} \sum_{j=0}^{r-1} \nu_j^{(i)} = 0\}\]

Fix \(r, d, \mathbb{C}^{nr-1} \ni \nabla \in \mathcal{N}_{r}^{n}(d)\)

Define \((E, \nabla, \{\ell_i^{(1)}\}_{i=1}^{n})\)

\(\nabla - \text{parabolic connection of rank } r \text{ degree } d\)
\( \Omega \) is an algebra v.s. of rank \( r \) and degree \( d \)

(2) \( \mathcal{F}: E \to E \otimes \mathcal{O}_C(D(t)) \)

(3) Decreasing filtration

\( \mathcal{F}_k = E \supset E_0 \supset E_1 \supset \ldots \supset E_k \)

\( \mathcal{F}_0 / \mathcal{F}_1 \) is isomorophic for \( \gamma \)

**Example**

\[
\begin{align*}
E|_{t_1} &= \mathbb{A}^2 \\
\text{res}_{t_1}(\mathcal{F}) &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{align*}
\]

\[
\left\{ \mathcal{F}_k: \mathbb{A}^2 \supset E_0 \supset E_1 \supset E_2 = \{0\} \right\} \cong \mathbb{A}^1
\]

\( \gamma \) choice of line \( l_1 \)

**Ex.**

\[
E|_{t_1} = \mathbb{A}^2
\]

\[
\text{res}_{t_1}(\mathcal{F}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
(A - 0 \times 0)(l_1) \subset l_2
\]

\[
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}(y) = (y)
\]

\[
\Lambda(l_1) \cap \{0\}
\]

\[
\frac{d}{dx} \text{ker } A = \langle (1) \rangle \quad \text{no module}
\]
Subbundle $F$

$(E, D) \not\subset (F, D_F)$

Vector bundle theory \[
\frac{\text{deg } F}{\text{rank } F} = \frac{\text{deg } E}{\text{rank } E}
\]

Poincaré polynomial

\[
\chi(E(m)) = \sum_{F \text{ ample}} E \otimes H \otimes F
\]

\[
= \dim H^0(E(m)) - \dim H^2(E(m)) - \text{deg } E(m) - \text{rank } E(m)(g-1)
\]

\[
= \chi(c_1 + m) - r(g-1)
\]

\[
E = L_1 \otimes \cdots \otimes L_r
\]

Stability gives a way of choosing good objects
or a way of throwing away bad objects

\[
M^\alpha(C, \chi, (E, D), \{L_i}\}, \alpha - \text{stable parabolic connection}) / \text{isom}
\]

Thm (Inaba, Inaba-Iwasaki-Saito)

\[
M^\alpha(C, \chi, \mathbb{R}, n, c) \text{ is a smooth quasi-projective variety with holomorphic symplectic} \quad \alpha \text{ of dim}
\]

\[
2^2(g-1) + \text{rank}(c-2) + 2
\]
\[ r = \frac{1}{2} \times \frac{1}{2} \times (g-1) + 2 = 2g \]

\((E, D)\)

root 1 of \( \psi \) and \( \theta \) connection

\( \{E\} \cong \text{Jac}^d(C) \)

\( \text{Jac}(C)^d \times H^0(C, \Omega^1_C) \)

\((C, \tau) \in \mathcal{M}_{\leq g}^n\)

\( \omega \in \mathcal{W}_r^n(d) \)

\( T_{\mathcal{M}_{\leq g}^n} \rightarrow \mathcal{M}_{\leq g}^n \)

\( C_{\leq g}^n \)

\( \varepsilon_{(k)} \uparrow \downarrow \)

\( T_{\mathcal{M}_{\leq g}^n} \rightarrow \mathcal{M}_{\leq g}^n \)

\((C, \tau) \) fix \ V \ vary \ \mathcal{W} \in \mathcal{W}_r^n(d) \)

\( \mathcal{W} = \{ \mathcal{W}^{(i)} \} \)

\( \{ \mathcal{W}^{(i)} \}_{0 \leq i \leq r-1} \) distinct parabolic structure

\( \mathbb{C}^{nr-1} \cong \mathcal{W}_r^n(d) = (\forall) > \mathcal{W}_r^n(d)^0 \)
$$(E, \nabla)$$ logarithmic connection on $G$ with singularities at $t = \{t_1, \ldots, t_n\}$

$C \setminus \{t_1, \ldots, t_n\} = U$ open set

Let us trivialize $E$ on $U$

$$E|_U \cong \mathbb{C}^r \times U$$

\[
\begin{pmatrix}
\frac{f_1'(z)}{f_1(z)} \\
\frac{f_2'(z)}{f_2(z)} \\
\vdots \\
\frac{f_r'(z)}{f_r(z)}
\end{pmatrix} dz + A(z) \begin{pmatrix}
\frac{f_1'(z)}{f_1(z)} \\
\frac{f_2'(z)}{f_2(z)} \\
\vdots \\
\frac{f_r'(z)}{f_r(z)}
\end{pmatrix} dz
\]

$$A(z) = \begin{pmatrix}
g_{11}(z) & \cdots & \cdots \\
\vdots & \ddots & \vdots \\
\vdots & \cdots & g_{rr}(z)
\end{pmatrix}$$

$$\nabla t(z) = 0$$

$$\therefore \quad \frac{d}{dz} \begin{pmatrix}
\frac{f_1'(z)}{f_1(z)} \\
\frac{f_2'(z)}{f_2(z)} \\
\vdots \\
\frac{f_r'(z)}{f_r(z)}
\end{pmatrix} = A(z) \begin{pmatrix}
\frac{f_1'(z)}{f_1(z)} \\
\frac{f_2'(z)}{f_2(z)} \\
\vdots \\
\frac{f_r'(z)}{f_r(z)}
\end{pmatrix}$$

ODE

$$f(z), \ldots, g_r(z)$$

linear independent solution for $\nabla t = 0$
\[ \Phi = \begin{pmatrix} g_1 & g_2 & \cdots & g_r \end{pmatrix} \]  

Vector function

\[ \mathbf{\Xi}(\mathbf{r}) = \begin{pmatrix} \vdots \end{pmatrix} \]  

\[ r \times r \text{ matrix} \]

Fundamental solution of \( \nabla = 0 \)

Over \( U \), you have \( \mathbf{\Xi}(\mathbf{r}) = (g_1, \ldots, g_r) \)

\( r \times r \) matrices \( \mathbf{F} \nabla \mathbf{\Xi}(\mathbf{r}) = 0 \)

Take analytic continuation of \( \mathbf{\Xi}(\mathbf{r}) \)

along the paths in \( \mathbb{C} \setminus \{t_1, \ldots, t_n\} \)

\[ \nabla \cdot \mathbf{\Xi}(\mathbf{r}) \]
\[ \frac{d}{dt} (\overline{\theta}(t) \overline{\varphi}(t)^{-1}) \bigg|_{t=0} = 0 \quad (\text{Exercise}) \]

\[ \forall M_{g} \in \text{GL}(r, \mathbb{C}) \]

\[ \frac{d}{dt} (\theta(t) \varphi(t)^{-1}) = M_{g} \]

\[ \theta(0) = M_{g} \theta(0) \]

\[ \text{homotopy class of } \gamma \]

Representation of \( \pi_{1}(\mathbb{C}\setminus \{t_{1}, \ldots, t_{r}\}, *) \):

\[ \Delta : \pi_{1}(\mathbb{C}\setminus \{t_{1}, \ldots, t_{r}\}, *) \to \text{GL}_{r}(\mathbb{C}) \]

\[ \gamma \quad \xrightarrow{\Delta} \quad \Delta(\gamma) = M_{g} \]

\[ (E, D) \quad \xrightarrow{\sim} \quad [\sigma] \in \text{Hom}(\pi_{1}(\mathbb{C}\setminus \{t_{1}, \ldots, t_{r}\}, \text{GL}_{r}(\mathbb{C})), \mathcal{M}(\mathbb{C})) \]

\[ M_{g} \mapsto \sigma(M_{g})^{-1} \]
\[ \frac{d}{d\bar{z}} \bar{A}(z) = A(z) \bar{A}(z) \]

\[ \frac{d}{d\bar{z}} \bar{A}(z) = A(z) \bar{A}(z) \]

\[ \bar{A}(z) \cdot \bar{A}(z) \]  \[ \cdot (\bar{z} \cdot \bar{z} - 1) = 0 \]

\[ \bar{A}'(z) + \bar{A}(z)\bar{A}(z) = 0 \]

\[ (\bar{z} \cdot \bar{z} - 1)' = -\bar{z} \cdot \bar{z} A(z) \bar{A}(z) \]

\[ \frac{d}{d\bar{z}} \left( \bar{A}(z) \cdot \bar{A}(z) \right)' = \bar{A}(z)' \cdot \bar{A}(z) + \bar{A}(z) \cdot \bar{A}(z) \]

\[ = A(z) \bar{A}(z) + A(z) \bar{A}(z) - A(z) \bar{A}(z) \bar{A}(z) \bar{A}(z) \]

\[ = A(z) \bar{A}(z) \bar{A}(z) \bar{A}(z) \bar{A}(z) \]

\[ = A(z) \bar{A}(z) \bar{A}(z) \bar{A}(z) \bar{A}(z) \]

\[ = A(z) \bar{A}(z) \bar{A}(z) \bar{A}(z) \bar{A}(z) \]

\[ \left[ A, \bar{A} \bar{A}^{-1} \right] \text{ does not work} \]

\[ \frac{d}{d\bar{z}} \left( \bar{A}(z) \cdot \bar{A}(z) \right) = 0 \]

\[ \Rightarrow \bar{A}'(z) = M \bar{z} \Rightarrow M \text{ constant} \]
\[ \mathbb{R}P_r \begin{pmatrix} C; c \end{pmatrix} = \text{Hom} \left( \pi_1 \left( C \setminus \{t_1, \ldots, t_n\} \right), \ast \right), \frac{\text{GL}(r; \mathbb{C})}{\text{Aff}(r; \mathbb{C})} \]

\[ \pi_1 \left( C \setminus \{t_1, \ldots, t_n\} \right) \text{ is generated by} \]
\[ \begin{array}{l}
\alpha_1, \ldots, \alpha_n \\
\beta_1, \ldots, \beta_n
\end{array} \]

\[ \pi_1 \left( C; c^+ \right) \]

\[ (\frac{g}{11} \begin{array}{c}
\alpha_i \beta_i \alpha_i^{-1} \\
\gamma_i \\
\delta_i
\end{array} \frac{g}{g^{-1}} \begin{array}{c}
\alpha_i \beta_i \\
\gamma_i \\
\delta_i
\end{array} ) = 1 \]

\[ \rho \in \text{Hom} \left( \pi_1, \text{GL}_r \mathbb{C} \right) \]
\[ \rho(\alpha_i) = A_i \in \text{GL}_r \mathbb{C} \]
\[ \rho(\beta_i) = B_i \]
\[ \rho(\gamma_i) = M_i \]
\[ \mathbb{R} P_{c+h}^r = \text{Hom}(\pi_1(C \setminus \{ \mathcal{A}_1, \ldots, \mathcal{A}_n \}), \mathbb{C}) \cong \{ (\mathcal{A}_1, \mathcal{B}_1, \ldots, \mathcal{B}_n, M_1, \ldots, M_n) \in \text{GL}(r, \mathbb{C}) \mid \prod_{i=1}^{n} [\mathcal{A}_i, \mathcal{B}_i] M_i M_{i+1} \cdots M_n = I \} \]

\[ n > 1 \]

\[ \cong \text{GL}(r, \mathbb{C})^{\otimes n-1} \]

\[ M_n = (\prod [\mathcal{A}_i, \mathcal{B}_i] M_i, \ldots, M_{n-1})^{-1} \]

affine variety

\[ \text{GL}(n, \mathbb{C}) = \{ x = (x_1), \det(x) \neq 0 \} \]

Spec \( \mathbb{C}[x_i, \det(x)^{-1}] \) affine variety

\[ \mathbb{R} P_{c+h}^r \cong \text{Spec} \mathbb{R}_{2g+n-1} \] R affine coordinate ring of

\[ \text{GL}(n, \mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R} \]

\[ \phi(A_i) \]

\[ (pA_i p_i, pB_i p_i', \ldots) \]
$GL(n, \mathbb{C})$ acts on $\mathbb{P}^{2g+n-1}$ naturally.

Reductive group $GL(r, \mathbb{C})$ ring of invariants of $\mathbb{P}^{2g+n-1}$ by the action

$$\frac{\text{Tr}(A_i)}{\text{Tr}(A_i A_j)} \in \mathbb{C}[x_1, \ldots, x_N] / \mathbb{I}$$

Hilbert fundamental theorem says that $\mathbb{C}[x_1, \ldots, x_N] / \mathbb{I}$ is a finitely generated ring.

Commutative ring

$$RP^r(\mathcal{C}) = \text{Hom}(\pi_1(\mathbb{C}\setminus \{t_1, \ldots, t_n\}), \mathbb{C}) = \text{Spec}(R_{g+n-1}) \subset \text{affine variety} \mathbb{C} \subset \mathbb{C}^N$$
\[
\text{Spec } \big( \mathbb{P} \frac{\mathbb{C}(t, C)}{\mathfrak{g}_{\mathfrak{g} + \mathfrak{u} - 1}} \big)
\]

\[M_1, M_2, \ldots, M_n\]

\[\rho(\varphi(t)) \, \rho(t) \, \rho(t_n)\]

\[\chi(M_1, x) = \det(xI_n - M_1)\]

\[
\left( d + \frac{\chi^{(i)}}{t_i} \right) f = 0
\]

\[f = \exp(-\chi^{(i)}(t))\]

\[
\frac{r-1}{1} \sum_{\gamma=0}^{r-1} \left( x - \exp(-2\pi i (1 - \gamma^{(i)})) \right)
\]

\[= x^r + a_1^{(i)} x^{r-1} + \ldots + a_0^{(i)}\]
\[ A = \text{Spec } \mathbb{C}[a^{(i)}_d] \]

\[ \mathbb{R}P^r_{(c, \mathbb{R})} \rightarrow \mathbb{R}P^r_{(c, \mathbb{C})} \]

\[ \mathcal{A}^r \rightarrow a \]

\[ \mathcal{T}' \rightarrow T_{\text{can}} \rightarrow \mathcal{M}_{\text{can}} \quad (C, \mathbb{C}) \]

universal
covering

highly transcendental

\[ \mathcal{M}^{\mathcal{G}}(c_{j, n}, 0) \rightarrow \mathbb{R}H \rightarrow \mathcal{T}' \times \mathbb{R}P^r \quad \]

\[ \downarrow \quad (\varepsilon, \mathcal{D}, (\ell_i)) \]

\[ \mathcal{T}' \times \mathcal{N} \rightarrow \mathcal{T}' \times \mathcal{A}^r \quad \]

\[ \mathcal{D} = \left[ \text{Ker } \mathcal{D} \mid \mathcal{C}(t_1, \ldots, t_n) \right] \]

\[ \mathcal{R}H(c_{j, h})_{\mathbb{R}} \text{ gives an analytic resolution of singularities} \]
\textbf{Painlevé VII Case}

\[ C = \mathbb{P}^4, \quad S = 4 \]

\[
\begin{align*}
\text{rank} & \quad n = 2 \\
\text{degree} & \quad d = -1 \\
E & = \langle t_1, t_2, t_3, t_4 \rangle \\
n_i & = 0
\end{align*}
\]

Fixing the determinant \( (L, D) = (C^4 + t_4, d) \)

\[ \lambda = (\lambda_i)_{1 \leq i \leq 4} \quad \text{one of the eigenvalues} \]

\[ \mathcal{H} = \{ t_1, t_2, t_3, t_4 \} \rightarrow \{ 0 \leq t_1, \infty \} \]

\[ \text{SL}(2,C) \circ (L, D) \]

\[ \mathcal{V} = \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_2 & \lambda_3 & \lambda_4 & \lambda_1 \\
\lambda_3 & \lambda_4 & \lambda_1 & \lambda_2 \\
\lambda_4 & \lambda_1 & \lambda_2 & \lambda_3
\end{pmatrix} \]

\[ \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathbb{C}^4 \]

\[ M^0_{(4, 2, -1)}(\lambda, 2, -1) \quad 2\text{-dim} \]

\[ = S^0_{(3, \lambda)} \]
Family of affine cubic surfaces of Freie-Klein.
Jimbo and Tsuchioka.

Fix \( a = (a_1, a_2, a_3, a_4) \in \mathbb{C}^4 \) 

\[
\text{Hom}(\mathbb{P}^4, \{t_1, t_2, t_3, t_4\}^*), \ SL(2, \mathbb{C})
\]

\[
\text{SL}_2(\mathbb{C})^3 \cong \text{Aff}(\mathbb{C}^2)
\]

\[
\text{Spec}(R)
\]

\[
R = \mathbb{C}[x_1, x_2, x_3, a_1, a_2, a_3, a_4]
\]

\[
M_4 = (M_1 M_2 M_3)^{-1}
\]

\[
a_i = \text{Tr}(M_i) \quad 1 \leq i \leq 4
\]

\[
x_1 = \text{Tr}(M_2 M_1)
\]

\[
x_2 = \text{Tr}(M_1 M_3)
\]

\[
x_3 = \text{Tr}(M_1 M_2)
\]

\[
I = \langle f \rangle
\]

\[
f : (x_1, x_2, x_3) \mapsto x_1 x_2 x_3 + x_1^2 x_2 + x_2^2 x_3
\]

\[
-\theta(a) x_1 - \theta_2(a) x_2 - \theta_3(a) x_3 + \theta_4(a)
\]

\[
\theta_i(a) = \theta_1(a) = a_1 a_2 a_3 a_4 + a_0^2 a_1 a_2 a_3 + a_0^3
\]
\( \mathbb{A} = \text{Spec}(R) \subset \mathbb{C}^7 \)

\[ f(x, a) = 0 \]

\( \mathbb{A} = \mathbb{C}^4 \approx \text{Spec} \mathbb{Q}[a_1, a_2, a_3, a_4] \)

\[ \mathcal{X} = \{ (x_1, x_2, x_3, x_4) \in \mathbb{A} \} \quad \text{subject to} \quad f(x, a) = 0 \]

affine abe surface

\[
\begin{cases}
  f = 0 \\
  \frac{\partial f}{\partial x_i} = 0
\end{cases}
\]

Gröbner to determine if singularities

\( \mathbb{R}^{+} \)

correspondence

\[ a_1 = 2 \cos 2\pi x_i \]

\[ a_i = \exp(-2\pi x_i) \]

\[ \exp(2\pi i x_i) \]
\[ \begin{align*}
\mathcal{U} S^0_{r,t} & \cong S^0_{\text{aff}} \\
\text{have all fibers are smooth} & \\
& \text{proper biholomorphic} \\
& \text{analytic res of singularities} \\
\end{align*} \]

\[ a_i - 2 \\
\lambda_i = 0 \\
\mathbf{D} = \mathbf{D}^+ + \frac{1}{\nu - 1} \left( \begin{array}{ccc} 0 & k & \cdots \\
0 & 0 & \ddots 
\end{array} \right) + \cdots \]

\[ M \quad k = 0 \quad \text{Parabolic structure} \]

\[ P^1 \quad \text{Singular pt} \]

affine variety

can't contain compact hol curve except pt
$a = (2, 2, 2, 2)$

$\lambda = (0, 0, 0, \frac{1}{2})$

$Y^0$ negative section

$Y$ positive section

Elliptic curve $H^1(Y, \Theta) = 0$

$H^2(Y, \mathbb{Z}) \cong \text{Pic}^0 (Y) = \mathbb{Z} Y_0 + \mathbb{Z} F$

\[
\begin{cases}
Y_0^2 = -2 \\
Y_0 \cdot F_x = 4 \\
F^2 = 0
\end{cases}
\]

$Y \sim Y_0 + 2F$

Linear equivalent

$Y^2 = (-2 + 4 + 0) = 2$

$Y$ can move

$Y \cong \mathbb{P}^1$

$Y \cdot y' = 2$

Elliptic curve $C.F > 2$
Canonical bundle

\[ K_{F_2} \in \text{Pic}(F_2) \]
\[ = aY_0 + bF \]

Adjunction formula

\[ K_{Y_0} = \left( K_{F_2} + Y_0 \right) \bigg|_{Y_0} \]

\[ \deg K_{Y_0} = K_{F_2} \cdot Y + Y_0^2 \]
\[ -2 = -2 = (K_{F_2} \cdot Y_0) - 2 \]
\[ -2 = -2a + b \]
\[ \Rightarrow b = 2a \]

\[ \deg K_F = (K_{F_2} + F) \cdot F \]
\[ -2 = a + F^2 = 0 \]
\[ a = -2 \]

\[ K_F = -2Y_0 - 4F \]

\[ C \sim 2Y \sim 2Y_0 + 4F \]

\[ K_C = (K_{F_2} + C) / C \]
\[ = O_C = O_C \Rightarrow C \text{ elliptic curve} \]
Elliptic fibration

$S' \xrightarrow{f} \mathbb{P}^1$

How to blow up $S'$

$C \sim 2Y \sim 2Y_0 + 4F$

$\uparrow$ pos elliptic curve

Blow-up intersection

Famous degenerate fiber of Kodaira

$I_0 \ast D^{(1)}_4$

Another $\phi_Y$

Elliptic curve passing through 17 points must pass through 9th point
$S' \xrightarrow{\text{hyperK"ahler}} S$

relation of $p^2$

\[6 I_1 + I_0^* - 12 \text{ topological Euler number}\]
Relation to Painlevé equations

Classical Painlevé equations

\((\ast)\) \quad x'' = R(t, x, x') \quad \text{ODE}

\[ \frac{d}{dt} \quad x = x(t) \]

\[ R(t, x_0, x_1) \in C(t, x_0, x_1) \]

Def. The ODE \((\ast)\) has Painlevé property if \((\ast)\) has no movable singularities other than poles

\((t, x_0)\)

\((t, s, c_1)\) initial values

Solve ODE \((\ast)\) with initial values \((t_0, c_0, c_1)\)

\[ x = \Phi(t, t_0, c_0, c_1) \text{ satisfying } (\ast) \] and

\[ x(t_0) = c_0, \quad x'(t_0) = c_1 \]

\[ x(t) = \Phi(t, t_0, c_0, c_1) \text{ holomorphic function near } t = t_0 \]

Take analytic continuation of this solution \( x(t) = \Phi(t, t_0, c_0, c_1) \)

\( (\ast) \) Then \( x(t) \) might have some singularity

This singularity of \( x(t) \) may depend on the choice of initial condition \((t_0, c_0, c_1)\)
If a singularity \( x(t) \) depends on \( (t_0, c_0, c_1) \) then the singularity of soln \( x(t) \) is called \textit{movable}.

\[
x' = \frac{1}{t-a}
\]

\( \Rightarrow \) \( x(t) = \log(t-a) + C \)

\( (t_0, c_0) \) initial value

\[
x(t_0) = c_0
\]

\[
c_0 = \log(t_0-a) + C
\]

\[
c = c_0 - \log(t_0-a)
\]

\( t=a \) singularity

\( t=a \) \textit{Not movable}

\textit{Non movable singularity}

\[
(x')^2 = 4x^3 - g_2x^2 - g_3
\]

\( g_2, g_3 \in \mathbb{C} \)

The sol. are given by \textit{Weierstrass} \( P \)-function \( P(t) \)

\[
P(t) = \sum_{(\omega_1, \omega_2) \in \mathbb{Z}^2 \backslash \{0\}} \frac{1}{(t-\omega_1-\omega_2)^2} - \frac{1}{(\omega_1+\omega_2)^2}
\]

\[
+ \frac{1}{t^2}
\]

\( \omega_1, \omega_2 \) \textit{periods of elliptic curve}

\[
y^2 = 4x^3 - g_2x - g_3
\]
\[(P(t)')^2 = 4P(t)^3 - g_2 P(t) - g_3\]

\[t \rightarrow t + b \quad t \rightarrow t - b\]

\[(P(t-b)')^2 = 4P(t-b)^3 - g_2 P(t-b) - g_3\]

initial condition impose

\[P(t_0 - b) = c_0\]

\[b \text{ is determined by this condition}\]

\[x(t) = P(t-b)\]

pole of order 2 at \( t = b \) (mod \( \Delta v_1 + \Delta v_2 \))

depends on \((t_0, c_0)\)

movable singularity

pole of order 2

Ex

\[\int \frac{dx}{x^3} = \int \frac{dt}{x^b}\]

\[-\frac{1}{2} x^{-2} = t + c\]
Ex.
\[ x' = x^{-1} \]
\[ \int x \, dx = \int dt \]
\[ \frac{1}{2} x^2 = t + C \]
\[ x^2 = 2(t + C) \]
\[ x = \sqrt{2(t + C)} \quad \text{branching point at } t = -C \]
\[ C_0 \to c = \sqrt{2(t_0 + C)} \quad \text{c depends on } (t_0, c) \]

Does not satisfy Painlevé property

Then (Painlevé)

First order ODE which satisfy Painlevé property is one of the following

1) Ricatti equation
\[ x' = a(t) x^2 + b(t) x(t) + c(t) \]
\[ a, b, c \in C(t) \]

2) Weierstrass equation
\[ (x')^2 = 4x^3 - g_2 x + g_3 \]
Thm (Painlevé Gambier) 1904?

If ODE

\[ x'''(t) = P(t, x, x') \]

satisfies Painlevé property

then it can be classified in the following table

\[ P_J : \quad J = 1, \ldots, 6 \]

\[ P_1 : \quad x''' = 6x^2 + t \]

Painlevé equations can be obtained from isomonodromic deformation of linear equations

\[ \begin{array}{c}
\text{PV} \\
P^4 = C \cup \{00\} \\
\end{array} \]

\[ D = \Re a \left( \frac{d}{dz} + \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_\infty}{z^2} \right) \, dz \]

on \[ E = C_{p4} \cup C_{p1} \]

\[ \{q, \theta_0, \theta_\infty, \theta, t\} \]

parameters

Singularity at

Katz invariant \[ D = 0, 1, \infty \]

\[ \begin{array}{c}
\quad \downarrow \\
\text{regular sing} \\
\quad \uparrow \\
\text{poles of } \, D \text{ at } \infty \text{ is order 2}
\end{array} \]
\( \partial_0, \partial_1 \) local asymptote

\[ \mathcal{D} = d + A(z, p, t^2, t) \]

\[ \mathcal{D}_d = \frac{d}{dz} + A(z, p, t^2, t) \]

\[ \mathcal{D}_t = \frac{d}{dt} + A(z, p, t^2, t) \]

\( \mathcal{D} \) defined on \( \mathbb{P}^4 \times \mathbb{C}/\{0, 1\} \)

Flatness of \( \mathcal{D} \)

Curvature of \( \mathcal{D} = 0 \)

\[ \mathcal{D} = d + A dz + B dt \]

\[ \mathcal{D} \circ \mathcal{D} = (d + A dz + B dt) \cdot (d + A dz + B dt) \]

\[ = d^2 + dA dz + dB dt + A dB dz + B dA dz + B dB dt \]

\[ = \left[ -\frac{d^4}{d t^4} + dA + (AB - BA) \right] dz \cdot dt \]
\[ \frac{dA}{dt} = dB + [A, B] \begin{align*}
\implies & \text{ Zero curvature equation} \\
\implies & \text{ Isomonodromic equation}
\end{align*}

By solving (t), you can obtain
\[ \frac{dl}{dt} = \frac{2p}{t}, \quad \frac{dq}{dt} = \_ \_ \_ \_ \_ \_ \_ \]

\((P, q)\) parameterize the connection
\[ \nabla = d + A \text{d}z \]
with fixing \( \Theta \) (local exponent)

\((P, q)\) is the coordinate of moduli space of connection with fixed type of singularity at \( z = 0, 1, \infty \)
fixing \( \Theta \)

\[ (p(t), q(t)) \]

Hamiltonian depends on \( t \) \quad Non-autonomous Hamiltonian system
Garner, Okamoto, Jimbo, Miwa-V
van der Put - Saito
(1) Review from tamely ramified case

\[ G = GL(r, C) \]

\[ C: \text{a non-singular projective curve of genus } g \]

\[ \mathfrak{t} = \{ t_1, \ldots, t_n \} \quad t_i \in C \quad i \neq j \]

\[ d \in \mathbb{Z} \]

\[ \nu = \{ \nu(i) \} \quad 0 \leq j < r-1 \quad \in \mathbb{C}^r \]

\[ M_{r,n}(d) = \left\{ \sum_{i=0}^{n} \sum_{j=0}^{r-1} \nu(i) = -d \right\} \]

weight \[ \alpha = \{ \alpha_i^{(j)} \} \quad 1 \leq i \leq n \]

\[ 0 < \alpha_1^{(1)} < \alpha_2^{(1)} < \ldots < \alpha_r^{(1)} < 1 \]

\[ M^\alpha_{(C, \mathfrak{t})} (\nu, r, n, d) \cap \{ (E, \rho \rho_0^{(1)}) \mid 1 \leq i \leq n \}
\]

\[ D: E \rightarrow E \otimes S_C (ht) \]

\[ \text{deg } E = d, \text{ out for } \alpha \text{-stable } \mathbb{Z} \text{-parabolic} \]

\[ \text{Connection on } C \]

Thus \[ n \gg 1 \quad \alpha \text{ generic} \]

\[ M^\alpha_{(C, \mathfrak{t})} (\nu, r, n, d) \text{ is a smooth } \]

quasi-projective variety of dim

\[ 2r^2(q-1) + nr(r-1) + 2 \]

3. \[ \Omega: \text{nondegenerate symplectic form on } M^\alpha_{(C, \mathfrak{t})} \]
\[ \mathcal{M}_{g,n} \times \mathcal{M}_r^n(d) \]

\[ \mathcal{M}^a(r, n, d) \xrightarrow{\pi} \mathcal{M}^a(r, n, d) \]

\( \pi \) smooth morphism

\( \mathcal{M}_{g,n} \times \mathcal{M}_r^n(d) \xrightarrow{\pi} \) rel proj morphism

\( ((C, \pi), \nabla) \leftarrow \)

\[
\mathcal{M}^a \left( C, \pi, \lambda \right) = \left\{ \left( E_1, E_2, \phi, \nabla, \{ \ell \} \right) \right\}
\]

\( \lambda \)-stable \( \nabla \) parabolic connection

\( E_1, E_2 \) rank \( r \) bundles over \( C \)

\( \nabla : E_1 \to E_2 : \mathcal{O}_C \)-linear \( \phi(\sigma) = a \phi(\sigma) \)

\( \nabla(\sigma) = d\alpha \otimes \phi(\tau) + a\varphi \)

\( \phi \)-twisted Leibnitz rule

\( a \in \mathcal{O}_C, \sigma \in \mathcal{E} \) projective

\[
\mathcal{M}^a \left( C, \pi, \lambda \right) \xrightarrow{\psi} \mathcal{M}^a \left( C, \pi, \lambda \right)
\]

\[
( E, \nabla, l ) \leftrightarrow ( E, E, \text{id}_E, \nabla, \{ l \} )
\]

Generalization of \( \lambda \)-connection
Module of rep of $\pi_1$

\[
\pi_2 = \pi_1 (C \setminus \{t_1, \ldots, t_n\}, *)
\]

\[
= \langle a_1, b_1, \ldots, d_g, \{g_i \} ; r_1, \ldots, r_n \rangle
\]

$n \geq 1$

\[
= \langle a_1, \ldots, b_g, r_1, \ldots, r_n \rangle \text{ free group}
\]

\[
\text{Rep}(\pi_2, GL(n, C)) \overset{\text{Categorical Quotient}}{\longrightarrow} \text{Hom}(\pi_2, GL(n, C)) / \text{Ad}(GL(n, C))
\]

\[
GL^{2g}(\mathbb{C}) + (n-1) \quad \quad p \in GL(n, C)
\]

\[
(A_1, A_2, \ldots, A_g, B_1, \ldots, B_g, N_1, \ldots, N_{n-1})
\]

\[
\mapsto (PA P^{-1}, \ldots, PB P^{-1}, \ldots, PM P^{-1}, \ldots)
\]

$R =$ Affine coordinate ring of $GL(r, \mathbb{C})^{2g^2 + (n-1)}$
$GL(n, C) \cong R$ by adjoint action

$\text{Ad}(GL(n, C))$ ring of invariants

$\text{Tr}(A_0^i) = \text{Tr}(A_i^j A_j)$

Finitely generated ring (Hilbert)

$\text{Spec} (\text{Ad}(GL(n, C))) \cong \text{Irreducible Affine variety}$ (Separated Hausdorff)

(Thm (Simpson))

Let $\text{Rep}(\pi_A, GL(n, C))$

$= \text{Spec} \text{Ad}(GL(n, C))$

$= \{ \text{Jordan equivalence class of }$

$\pi: \pi \rightarrow GL(n, C) \}$

$\pi = \pi_0 \oplus \pi_1 \oplus \ldots \oplus \pi_k$ sub rep

$\cong \text{Ind} (\pi_0/\pi_1) \oplus (\pi_1/\pi_2) \oplus \ldots \oplus \pi_k$

Jordan equivalent

$\cong \begin{pmatrix} \Box & \Box & \ast & \Box \\ \Box & \Box & \Box & \Box \\ \ast & \ast & \ast & \ast \\ \Box & \Box & \Box & \Box \end{pmatrix}$
Riemann–Hilbert correspondence

\[ M^\infty(r, n, d) \xrightarrow{\text{RH}} \text{Rep} \]

\[ M_{\text{g,n}} \times M^n_{\text{g,n}}(d) \xrightarrow{\text{rh}} M_{\text{g,n}} \times \mathbb{A}^n_{r}(d) \]

\[ (C, \pi) \xrightarrow{\mathcal{V}} (\bigvee_{i=1}^{n}) \]

\[ (C, \pi) \cdot \mathbb{A}^n_{r}(d) \]

\[ \pi_{2}^{-1}((C, \pi), \mathcal{V}) \rightarrow \pi_{2}^{-1}((C, \pi), \mathcal{V}) \]

\[ M^\infty_{C, \pi}(\mathcal{V}, r, n, d) \xrightarrow{\text{Rep}(G), GL(r, \mathbb{C})} \]

\[ \text{Jordan norm class of } \]

\[ \text{monodromy rep associated to } \nabla |_{C \setminus \{t_1, \ldots, t_n\}} \]

Thm (Inaba, Iwako, Inoue, Saito)

1. \( \mathcal{V} \) is a proper surjective holomorphic map

2. Generic \( \mathcal{V} \) is a holomorphic map
Remark:
\[ V = \text{a special} \]
\[ \kappa = \text{rh}(V) : \text{special} \]
\[ \text{Rep}(\pi_1(\text{GL}(r_1)))^{\kappa} \]
has singularities
\[ M^{\text{sing}}(C_{\text{rel}})(X, r_1, u, o) \]
non-singular variety
\[ \text{so } RH(C_{\text{rel}}), V \text{ gives an analytic resolution of singularities of } \text{Rep}(\pi_1, \text{GL}(r_1))^{\kappa} \]
Remark: You can never expect that RH is algebraic

Witten paper 0710. 0631

\[ M^{\text{sing}}(C_{\text{rel}})(X, r, u, o) \rightarrow \text{Rep}(\pi_1, \text{GL}(r_1, C))^{\kappa} \]
\[ \uparrow \text{iso complex structures} \]
\[ M^{\text{sing}}(C_{\text{rel}})(X, r, n, d) \]
PVI case and other $P^*_j$ ($j = 1, \ldots, 8$) of Painlevé Equations

All other of 8 types can be obtained by isomonodromic deformation of rank 2 connection with regular or irregular singularities

$C = \mathbb{P}^2$, $E = \mathcal{O} \oplus \mathcal{O}(-1)$

PVI

$\mathcal{H} = \{t_1, t_2, t_3, t_4\}$

$\nabla : E \rightarrow E \otimes \mathcal{S}^1_{\mathbb{P}^2}(t_1 + t_2 + t_3 + t_4)$

$M^x_{\mathcal{H}}(t, 2, 4) : 2$-dim smooth variety

$M^{\alpha}_{\mathcal{H}}(t, 2, 4)$ smooth projective surface

$\gamma_0^2 = -2$

\[ F_2 \rightarrow \mathbb{P}(\mathcal{O}_E \oplus \mathcal{O}(\gamma)) \]

\[ \mathbb{P}^1 \]

$SL(2, \mathbb{C}) \quad (\Lambda^2 \mathcal{E}, \Lambda^2 \nabla) = (\mathcal{O}(-1), \mathcal{O})$

$\mathcal{N} = \left\{ \left( \begin{array}{cccc} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 \end{array} \right) \right\}$ \quad $\text{res}_t(\mathcal{N})$
Blow up 8 pts \( \text{rk } \phi = 1 \)

\( \text{rk } \phi > 0 \)

Higgs bundle

\[
\leq \mathcal{M}_{ht} (\nu, 4, 2)
\]

\[
\cup
\]

\[
\mathcal{M}_{ht} (\nu, 4, 2)
\]

\( \text{rk } \phi = 2 \)

\[
E = O \oplus O(-1)
\]

\[
\phi : E_1 \longrightarrow E_2 \quad \text{homomorphism}
\]

\[
\begin{array}{c}
\phi
\end{array}
\]

\( \text{rk } \phi = 2 \)

\( \phi \) iso

\[
\eta = \phi D + A
\]

\[
\phi^{-1} D = id + A^{-1}A
\]

Compactification has a stratification by rank of \( \phi \)
\[ \overline{M}_{\text{ht}}(2,4,?) \rightarrow \overline{M} \text{ Higgs} \]

\[ \text{Hitchin fibration} \]

\[ \overline{B} = \mathbb{P}^1 \]

\[ B = C \]

\[ \infty \text{ fiber} \]

\[ \mathbb{P}^1 \cup \overline{\mathbb{P}^1} \text{ interestingly } \mathbb{P}^1 \]

Kodaira I_{\infty}^* type

Can prove no elliptic fibration in \( \overline{M}_{\text{ht}}(2,4,2) \)

3 no non-constant alg. function in \( M_{\text{ht}}(2,4,2) \)
\[ \text{Rep}_+ = \text{Hom}(\pi_1(\mathbb{P}^4 \setminus \{1, 2, 3\}), \mathbb{C}^N) / \text{Ad} \]

\[ X_6 \subset \mathbb{C}^7 \]

\[ x_1 x_2 x_3 x_4 \]

\[ x_1 x_2 x_3 + x_1 x_2^2 + x_3^2 + \ldots \]

\[ \mathbb{C}^4 \ni (a_1, a_2, a_3, a_4) = \alpha \]

\[ a_i = \frac{2 \cos(-\pi \sqrt{-1} x_i)}{2 \cos(-\pi x_i)} \]

\[ \mathbb{C}^3 \cup \overline{\mathbb{C}} \text{ cubic} \]

\[ \mathbb{P}^3 \to \mathbb{C} \to \overline{\mathbb{C}} \]

\[ \mathbb{P}^3 \to \overline{\mathbb{C}} \leftarrow \overline{\mathbb{C}} \text{ smooth projective cubic} \]

\[ \overline{\mathbb{C}} \setminus \mathbb{C} \]

\[ \mathbb{M}_{+}(\mathbb{P}, 4, 2) \to \text{generic} \]
$z$ special $\rightarrow$ a special

$M_+ \left( k \mathbb{Z}, 4, 2 \right) \quad $ Stein but not affine

$U$

$D$ exceptional divisor

PVI

PV

order of pole $\begin{array}{c} t_1 + t_2 + 2t_3 \\ 1 & 1 & 2 \end{array}$

$\mathcal{D} : E \rightarrow E \otimes \mathcal{O}(t_1 + t_2 + 2t_3)$

Marius van der Put, Saito

List up to 10 cases of each 2 case

w/ regular or irregular

$\Rightarrow$ Painlevé