Math 241
Some solutions for Homework 1

1.4.7a. First of all, let’s see what $\beta$ should be. In an equilibrium solution, the total amount of heat (between 0 and $L$) should be constant. But remember, we have

\[
\text{(Rate of change of total heat between 0 and } L) = \text{(Flux into } [0, L])
\]

\[
+ \text{(Total heat from sources in } [0, L])
\]

Now, the flux into $[0, L]$ is given by $\frac{\partial u}{\partial x}(L, t) - \frac{\partial u}{\partial x}(0, t) = \beta - 1$. And the heat source density (per unit length) is given by that extra '1' sitting in the PDE. So the total heat from sources is $\int_0^L 1 \, dx = L$. So for the total heat to be constant, we need

\[
\beta - 1 + L = 0,
\]

and so $\beta = 1 - L$.

Now let’s find the equilibrium value of $u$. We set $u_t = 0$ and find that $u_{xx} + 1 = 0$. Since $u$ doesn’t depend on $t$, this is basically an ODE of the kind you learned to solve in 240. Using whatever method you choose (integration, undetermined coefficients, etc.), we find

\[
u(x, t) = -\frac{x^2}{2} + c_1 x + c_2.
\]

Using the BC $u_x(0, t) = 1$, we find that $c_1 = 1$.

What is $c_2$? Here we need to use the initial condition $u(x, 0) = f(x)$ and the fact that the total energy will be constant, as we saw above. The total energy at time $t = 0$ is $\int_0^L f(x) \, dx$. The total energy at equilibrium is $\int_0^L \left(-\frac{x^2}{2} + x + c_2\right) \, dx$. Setting these two equal, we can solve for $c_2$, and get

\[
c_2 = \frac{L^2}{6} - \frac{L}{2} + \frac{1}{L} \int_0^L f(x) \, dx.
\]

2.3.2d. A lot of this stuff is the same as previous heat equation problems, so I’ll skip to the part where we have

\[
\phi'' + \lambda \phi = 0
\]

\[
G' + k\lambda G = 0.
\]

(Here we’re looking for product solutions $u(x, t) = \phi(x)G(t)$). The homogeneous boundary conditions will separate out to give $\phi(0) = 0, \phi'(0) = 0$. There are three cases in the equation for $\phi$:

(a) $\lambda > 0$. Then $\phi(x) = c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x)$. We have

\[
\phi(0) = c_2 = 0,
\]

so $c_2 = 0$, and

\[
\phi'(L) = c_1 \sqrt{\lambda} \cos(\sqrt{\lambda}L) = 0.
\]
We don’t want $c_1 = 0$, and we have $\lambda > 0$, so $\sqrt{\lambda} = 0$. So our only option is $\cos(\sqrt{\lambda}L) = 0$, which leaves us with

$$\sqrt{\lambda}L = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \ldots$$

Rearranging, this leaves us with

$$\lambda = \left(\frac{(2n - 1)\pi}{2L}\right)^2, \quad n = 1, 2, \ldots$$

and

$$\phi(x) = \sin\left(\frac{(2n+1)\pi}{2L}x\right).$$

(b) $\lambda = 0$. We have $\phi(x) = c_1 x + c_2$. But $\phi(0) = c_2 = 0$ and $\phi'(L) = c_1 = 0$, so we have no nonzero eigenfunctions.

(c) $\lambda < 0$. We have $\phi(x) = c_1 \sinh(\sqrt{-\lambda}x) + c_2 \cosh(\sqrt{-\lambda}x)$. Then $\phi(0) = c_2 = 0$, and $\phi'(L) = c_1 \sqrt{-\lambda} \cosh(\sqrt{-\lambda}L) = 0$. We won’t have $\sqrt{-\lambda} = 0$ or $\cosh(\sqrt{-\lambda}L) = 0$, so our only option is $c_1 = 0$. Thus we have no nonzero eigenfunctions.

2.3.3. In this situation, the solution will look like

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)e^{-\left(\frac{n\pi}{L}\right)^2 kt},$$

where the $B_n$ are given by the Fourier sine series coefficients of the ICs. In particular, if $u(x, 0) = f(x)$, then

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$  

(On the homework, unless otherwise specified, you generally don’t need to repeat any work you saw in class; you can just say “we did this in class”.)

b. Note that in this case, we can read off the sine series coefficients directly. So $B_1 = 3, B_3 = -1$, and all other $B_n = 0$. So

$$u(x, t) = 3 \sin\left(\frac{\pi}{L}x\right)e^{-(\frac{\pi}{L})^2 kt} - \sin\left(\frac{3\pi}{L}x\right)e^{-(\frac{3\pi}{L})^2 kt}.$$  

c. In this case, we can’t immediately see what $B_n$ is, so we have to use the integral:

$$B_n = \frac{2}{L} \int_0^L 2 \cos\left(\frac{3\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx.$$  

(Note that the problem said we didn’t have to do the integral.)

B.b. (Note that part a is basically 2.3.2d and/or the problem we did in class.)

Our solution will be

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{(2n-1)\pi}{2L}x\right)e^{-(\frac{(2n-1)\pi}{2L})^2 kt},$$
for some appropriate values of $c_n$. These will be given by the formula

$$
c_n = \frac{\int_0^L x \sin \left( \frac{(2n-1)\pi}{2L} x \right) \, dx}{\int_0^L x \sin^2 \left( \frac{(2n-1)\pi}{2L} x \right) \, dx} = 2 \frac{\int_0^L x \sin \left( \frac{(2n-1)\pi}{2L} x \right) \, dx}{\int_0^L x \sin \left( \frac{2\pi}{2L} x \right) \, dx}.
$$

At the risk of defeating the point of these solutions, I won’t do this computation out; it’s more important to do the right computation (although you should be able to do a 104-level problem).

2.3.7e. From looking at our notes and/or the previous parts of this problem, we see that

$$
u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi x}{L} \right) e^{-\left( \frac{n\pi}{L} \right)^2 kt},
$$

where

$$
A_0 = \frac{1}{L} \int_0^L f(x) \, dx
$$

and

$$
A_n = \frac{2}{L} \int_0^L f(x) \cos \left( \frac{n\pi x}{L} \right) \, dx \quad (n \geq 1).
$$

Letting $t \to \infty$, all of the $e^{-\left( \frac{n\pi}{L} \right)^2 kt}$ terms will approach 0 for $n \geq 1$. So $u(x,t)$ will approach $A_0$.

Now, back in the equilibrium section, we said that the equilibrium for an insulated rod should be a constant, and that constant should be $\frac{1}{L} \int_0^L f(x) \, dx$. But that’s exactly what $A_0$ is.