Math 241
Some solutions for Homework 2

A. (a) \( f(x) = x^2 \) is an even function, so \( b_n = 0 \) for all \( n \).
(b) If \( -\pi < x < \pi \), and \( f \) is continuous at \( x \), then the series will converge to \( f(x) \). But \( f(x) = x^2 \) is continuous at every \( x \), so the series will equal \( x^2 \) whenever \( -\pi < x < \pi \). In addition, if \( f \) is continuous at \( -\pi \) and \( \pi \), and \( f(\pi) = f(-\pi) \), then the series will converge at \( \pm \pi \) too. We can easily see that \( (-\pi)^2 = \pi^2 \), and \( f \) is continuous at \( \pm \pi \).

So the series will converge to \( x^2 \) when \( -\pi \leq x \leq \pi \).
(c) Notice that

\[
\sum_{n=0}^{\infty} a_n = a_0 + \sum_{n=1}^{\infty} a_n \cos(n \cdot 0).
\]

But the previous part tells us that \( a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) = x^2 \) for any value of \( x \) in \([-\pi, \pi]\).

Thus, \( a_0 + \sum_{n=1}^{\infty} a_n \cos(n \cdot 0) = 0 \). Similarly, plugging in \( x = \pi \) instead, we have

\[
\sum_{n=0}^{\infty} (-1)^n a_n = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi) = \pi^2.
\]
(d)

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 \, dx = \frac{1}{2\pi} \left[ \frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{\pi^2}{6}.
\]

For \( n \neq 0 \):

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(nx) \, dx
\]

\[
= \frac{1}{\pi} \left[ \frac{x^2 \sin(nx)}{n} \right]_{-\pi}^{\pi} - \frac{2}{n} \int_{-\pi}^{\pi} x \sin(nx) \, dx
\]

\[
= \frac{1}{\pi} \left[ \frac{2x \cos(nx)}{n^2} \right]_{-\pi}^{\pi} - \frac{2}{n^2} \int_{-\pi}^{\pi} \cos(nx) \, dx
\]

\[
= \frac{4 \cos(n\pi)}{n^2} - \frac{2 \sin(n\pi)}{n^3} \bigg|_{-\pi}^{\pi}
\]

\[
= (-1)^n \frac{4}{n^2}.
\]
(e) From part (c), we know that

\[ a_0 + \sum_{n=1}^{\infty} a_n = 0. \]

Plugging in the values of \( a_0 \) and \( a_n \) from part (d), we have

\[ \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} = 0. \]

Rearranging, we get

\[ \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = -\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} = \frac{\pi^2}{12}. \]

Also from part (c), we know that

\[ a_0 + \sum_{n=1}^{\infty} (-1)^n a_n = \pi^2; \]

again, plugging in the values from part (d), we get

\[ \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} = \pi^2, \]

and we can rearrange to get

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \]

2.5.2. (a) For an equilibrium, we need the total heat flux in to be 0. Here the heat flux in is \( 0 \) on the left, right, and bottom sides, and \( \int_0^L f(x) \, dx \) on the top. So we need \( \int_0^L f(x) \, dx = 0 \).

(b) I’ll skip some steps.

Separating variables \( u(x,y) = \phi(x)h(y) \) will eventually give us

\[ \phi'' + \lambda \phi = 0, \quad \phi'(0) = 0, \quad \phi'(L) = 0 \]

\[ h'' - \lambda h = 0, \quad h'(0) = 0. \]

The solution to the eigenvalue problem for \( \phi \) is \( \lambda = 0 \), with \( \phi(x) = 1 \), and \( \lambda = \left( \frac{n\pi}{L} \right)^2 \), with \( \phi(x) = \cos \left( \frac{n\pi}{L} x \right) \). Then we use these values of \( \lambda \) in the equation for \( h \), and we end up with product solutions:

\[ \lambda = 0 : \phi(x)h(y) = 1 \]

\[ \lambda = \left( \frac{n\pi}{L} \right)^2 : \phi(x)h(y) = \cos \left( \frac{n\pi}{L} x \right) \cosh \left( \frac{n\pi}{L} y \right). \]

So our general solution looks like

\[ u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos \left( \frac{n\pi}{L} x \right) \cosh \left( \frac{n\pi}{L} y \right). \]
Now let’s use our boundary conditions. We have

\[ u_y(x, H) = \sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sinh\left(\frac{n\pi}{L} H\right) \cos\left(\frac{n\pi}{L} x\right) = f(x). \]

Expanding \( f(x) \) in a cosine series, we have

\[ f(x) = C_0 + \sum_{n=1}^{\infty} C_n \cos\left(\frac{n\pi}{L} x\right), \]

where \( C_0 = \frac{1}{T} \int_0^L f(x) \, dx \), and \( C_n = \frac{2}{T} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) \, dx \). Comparing coefficients, we have

\[ C_0 = 0, \]

and

\[ C_n = \frac{n\pi}{L} A_n \sinh\left(\frac{n\pi}{L} H\right). \]

Thus, we can set

\[ A_n = \frac{C_nL}{n\pi \sinh\left(\frac{n\pi}{L} H\right)}, \]

but we need \( C_0 = 0 \), so \( \int_0^L f(x) \, dx = 0 \).

(c) Since the total heat flux in is 0, the total heat is constant. Thus, \( \iint u(x, y) \, dx \, dy = \iint g(x, y) \, dx \, dy \). To evaluate the left-hand side of this equation, we should plug in our formula for \( u(x, y) \). But first notice that \( \int_0^L \cos\left(\frac{n\pi}{L} x\right) \, dx = 0 \) for \( n \geq 1 \). So all of the \( \cos\left(\frac{n\pi}{L} x\right) \cosh\left(\frac{n\pi}{L} y\right) \) will integrate out to 0, and we will get

\[ \iint u(x, y) \, dx \, dy = \int_0^H \int_0^L A_0 \, dx \, dy + \sum_{n=1}^{\infty} A_n \int \int \cos\left(\frac{n\pi}{L} x\right) \cosh\left(\frac{n\pi}{L} y\right) \, dx \, dy \]

\[ = LH A_0. \]

So \( LH A_0 = \iint g(x, y) \, dx \, dy \); rearranging, we get \( A_0 = \frac{1}{LH} \iint g(x, y) \, dx \, dy \).

4.2.1. (a) Equilibrium happens when \( u_t = 0 \). So \( (u_E)_t = 0 \); taking another time derivative, we get \( (u_E)_{tt} = 0 \). So the equation we need to solve is

\[ T \frac{d^2 u_E}{dx^2} - g\rho_0(x) = 0, \]

with boundary conditions \( u_E(0) = 0 \) and \( u_E(L) = 0 \). The solution to the ODE is

\[ u_E(x) = \frac{g}{\bar{T}} R(x) + c_1 x + c_2, \]

where \( R(x) = \int_0^x \int_0^{x_1} \rho_0(x_2) \, dx_2 \, dx_1 \) is the second antiderivative of \( \rho_0(x) \); if \( \rho_0(x) \) is a constant, then \( R(x) = \frac{\rho_0}{2} x^2 \). Using the boundary conditions, we get

\[ u_E(x) = \frac{g}{\bar{T}} R(x) - \frac{gR(L)}{\bar{T}L} x. \]

(If \( \rho_0 \) is a constant, we just get \( u_E(x) = \frac{g\rho_0}{2\bar{T}} x^2 - \frac{g\rho_0L}{2\bar{T}} x \)).
(b) Suppose that $u$ satisfies the equation

$$u_{tt} = c^2 u_{xx} - g.$$ 

Note that $u_E$ also satisfies

$$(u_E)_{tt} = c^2 (u_E)_{xx} - g$$

(since both sides are 0). Subtracting the second equation from the first, we get

$$u_{tt} - (u_E)_{tt} = c^2 u_{xx} - c^2 (u_E)_{xx}.$$ 

If $v = u - u_E$, this just says that $v_{tt} = c^2 v_{xx}$. 