Math 241
Some solutions for Homework 3

5.3.3. We multiply the equation

$$\phi'' + \alpha(x)\phi' + [\lambda\beta(x) + \gamma(x)]\phi = 0$$

by $H(x)$:

$$H(x)\phi'' + H(x)\alpha(x)\phi' + [\lambda H(x)\beta(x) + H(x)\gamma(x)]\phi = 0.$$ We want this to look like the Sturm-Liouville form:

$$\frac{d}{dx}(p(x)\phi') + [\lambda\sigma(x) + q(x)]\phi = 0;$$

matching the coefficients, we want

$$p(x) = H(x)$$
$$p'(x) = H(x)\alpha(x)$$
$$\sigma(x) = H(x)\beta(x)$$
$$q(x) = H(x)\gamma(x).$$

We see that $\frac{p'}{p} = \alpha(x)$. Integrating both sides, we get $\ln(p(x)) = \int \alpha(x)\,dx$, so $p(x) = e^{\int \alpha(x)\,dx}$. Since $p(x) = H(x)$, we then have $\sigma(x) = e^{\int \alpha(x)\,dx} \beta(x)$ and $q(x) = e^{\int \alpha(x)\,dx} \gamma(x)$.

5.3.5. (a) The eigenvalues are $0, \left(\frac{n\pi}{L}\right)^2, \left(\frac{2n\pi}{L}\right)^2, \left(\frac{3n\pi}{L}\right)^2, \ldots$. Since we want to number the eigenvalues starting with $\lambda_1$, we set $\lambda_n = \left(\frac{(n-1)\pi}{L}\right)^2$, and the corresponding eigenfunction is $\phi_n(x) = \cos\left(\frac{(n-1)\pi}{L}x\right)$. The smallest eigenvalue is $\lambda_1 = 0$, and as $n$ increases, $\lambda_n$ will increase without maximum.

(b) We have $\phi_n(x) = \cos\left(\frac{(n-1)\pi}{L}x\right)$, whose zeroes are

$$\frac{L}{2(n-1)}, \frac{3L}{2(n-1)}, \frac{5L}{2(n-1)}, \ldots, \frac{(2n-3)L}{2(n-1)}.$$ That is, $\frac{mL}{2(n-1)}$ for $m = 1, 2, \ldots, n-1$. So there are $n-1$ zeroes.

(c) The completeness is just the fact that a function has a Fourier cosine series. The orthogonality is just the fact that

$$\int_0^L \cos\left(\frac{(m-1)\pi}{L}x\right) \cos\left(\frac{(n-1)\pi}{L}x\right) \,dx = 0$$

if $m \neq n$; we can verify this by directly evaluating the integral.

(d) The Rayleigh quotient is

$$\lambda = \frac{-p\phi|_a^b + \int_a^b [p(\phi')^2 - q\phi^2] \,dx}{\int_a^b \phi^2\sigma \,dx}.$$
In our case, \( p(x) = 1, q(x) = 0, \sigma(x) = 1, a = 0, \) and \( b = L. \) Since \( \phi'(0) = \phi'(L) = 0, \) we know that \(-\phi\phi'_0^L = 0. \) So the quotient just becomes

\[ \lambda = \frac{\int_0^L (\phi')^2 \, dx}{\int_0^L \phi^2 \, dx}. \]

Since \((\phi')^2\) and \(\phi^2\) are both nonnegative, both integrals must be nonnegative, so \(\lambda \geq 0,\) i.e. there are no negative eigenvalues.

Can \(\lambda = 0?\) If so, we’ll need \(\int_0^L (\phi')^2 \, dx,\) so then we need \(\phi'(x) = 0\) for all \(x.\) Thus \(\phi(x) = c\) must be a constant. But we can see this satisfies the problem and the BCs, so \(\lambda = 0\) is an eigenvalue.

5.9.1.b. We have \( \phi(x) \approx A(x)(c_1 \cos \psi(x) + c_2 \sin \psi(x)), \) where \( A(x) = (p\sigma)^{-1/4} \) and \( \psi(x) = \sqrt{\lambda} \int_a^x \sqrt{\frac{\sigma(x_0)}{\frac{\sigma(x_0)}{p(x_0)}}} \, dx_0. \)

I’ll state some important facts about these two functions: \( A(x) \) is never 0, \( A'(x) \) is relatively small, \( \psi(a) = 0, \) and \( \psi'(x) = \sqrt{\lambda} \sqrt{\sigma(x)} p(x). \)

In our case, \(a = 0\) and \(b = L.\) First, let’s make \(\psi(0) = 0.\) We have

\[ \phi(0) \approx A(0)(c_1 \cos \psi(0) + c_2 \sin \psi(0)) = c_1 A(0), \]

since \(\psi(0) = 0.\) So \(c_1 A(0) = 0.\) Since \(A(0) \neq 0,\) we need \(c_1 = 0.\) Now we use \(\phi'(L) = 0.\) We have

\[ \phi'(L) \approx c_2 A(L) \psi'(L) \cos \psi(L) + A'(L) \sin \psi(L) \]

\[ = c_2 A(L) \sqrt{\lambda} \sqrt{\sigma(L)} p(L) \cos \psi(L) + A'(L) \sin \psi(L). \]

Now, \(A'(L)\) is small compared to \(\sqrt{\lambda},\) so we can ignore the \(A'(L) \sin \psi(L)\) term. So

\[ \phi'(L) \approx c_2 A(L) \sqrt{\lambda} \sqrt{\sigma(L)} p(L) \cos \psi(L). \]

We want this last expression to be 0. What can be 0? To have a nonzero eigenfunction, we need \(c_2 \neq 0;\) \(\lambda\) is very large, so \(\sqrt{\lambda} \neq 0;\) and \(\sigma\) and \(p\) are both assumed to be positive. So we must have \(\cos \psi(L) = 0,\) and thus \(\psi(L) \approx (n + \frac{1}{2})\pi.\) Using our formula \(\psi(L) = \sqrt{\lambda} \int_0^L \sqrt{\frac{\sigma(x_0)}{p(x_0)}} \, dx_0,\) we get

\[ (n + \frac{1}{2})\pi \approx \sqrt{\lambda} \int_0^L \sqrt{\frac{\sigma(x_0)}{p(x_0)}} \, dx_0, \]

and we can rearrange to get

\[ \lambda \approx \left( \frac{n + \frac{1}{2}}{\int_0^L \sqrt{\frac{\sigma(x_0)}{p(x_0)}} \, dx_0} \right)^2. \]
7.7.2.d. Separating \( u(r, \theta, t) = \phi(r, \theta) h(t) = f(r) g(\theta) h(t) \) will eventually give us

\[
\begin{align*}
    h'' + \lambda c^2 h &= 0 \\
    g'' + \mu g &= 0 \\
    r \frac{d}{dr} (rf') + (\lambda r^2 - \mu) f &= 0.
\end{align*}
\]

with \( h(0) = 0 \), \( g(-\pi) = g(\pi) \), \( g'(-\pi) = g'(\pi) \), \( |f(0)| < \infty \), and \( f'(a) = 0 \). Solving for \( g \) first, we have \( \mu = \lambda^2 \), and \( g(\theta) = \cos(m \theta) \) or \( \sin(m \theta) \). Now let’s look at \( f \):

\[
    r \frac{d}{dr} (rf') + (\lambda r^2 - m^2) f = 0.
\]

Using the Rayleigh quotient for \( \phi(r, \theta) \) or just trying directly, we can see that \( \lambda < 0 \) is not possible, but \( \lambda = 0 \) is possible. If \( \lambda = 0 \), we have

\[
    r \frac{d}{dr} (rf') - m^2 f = 0,
\]

which is a Cauchy-Euler equation with solutions

\[
    f(r) = \begin{cases} 
        c_1 r^m + c_2 r^{-m} & \text{if } m > 0 \\
        c_1 + c_2 \ln r & \text{if } m = 0.
    \end{cases}
\]

We want \( |f(0)| < \infty \), so we set \( c_2 = 0 \) in either case. The boundary condition is \( f'(a) = 0 \), so we need \( c_1 ma^{m-1} = 0 \). If \( m > 0 \), this is impossible unless \( c_1 = 0 \), but for \( m = 1 \) we do get a solution: \( f(r) = 1 \).

If \( \lambda > 0 \), the solution is 

\[
    f(r) = c_1 J_m(\sqrt{\lambda} r) + c_2 Y_m(\sqrt{\lambda} r).
\]

(Note that this doesn’t work for \( \lambda = 0 \); we can’t do the substitution \( z = \sqrt{\lambda} r \). If \( \lambda < 0 \), then \( \sqrt{\lambda} \) is imaginary.) We want \( |f(0)| < \infty \), so we need \( c_2 = 0 \). The boundary condition \( f'(a) = 0 \) gives us \( c_1 \sqrt{\lambda} J'_m(\sqrt{\lambda} a) = 0 \). Let \( z_{m1}, z_{m2}, z_{m3}, \ldots \) be the positive zeroes of \( J'_m(z) \). Then we want \( \sqrt{\lambda} a = z_{mn} \), so \( \lambda_{mn} = (\frac{z_{mn}}{a})^2 \). One caveat: above we saw that if \( m = 0 \), then \( \lambda = 0 \) is possible, so we start with \( \lambda_{01} = 0 \), and let \( z_{02}, z_{03}, z_{04}, \ldots \) be the positive zeroes of \( J'_0(z) \), and then \( \lambda_{0n} = (\frac{z_{0n}}{a})^2 \) for \( n \geq 2 \).

In any case, solving for \( h \) now yields \( h(t) = \sin(\sqrt{\lambda_{mn}} t) \) if \( \lambda_{mn} > 0 \), and \( h(t) = t \) if \( \lambda_{mn} = 0 \), i.e. if \( m = 0, n = 1 \). Let’s set

\[
    h_{mn}(t) = \begin{cases} 
        t & \text{if } m = 0, n = 1 \\
        \sin(\sqrt{\lambda_{mn}} t) & \text{otherwise}
    \end{cases}
\]

and

\[
    f_{mn}(r) = \begin{cases} 
        1 & \text{if } m = 0, n = 1 \\
        J(\sqrt{\lambda_{mn}} r) & \text{otherwise}.
    \end{cases}
\]

Then the solution is

\[
    u(r, \theta, t) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} f_{mn}(r)(A_{mn} \cos m \theta + B_{mn} \sin m \theta) h_{mn}(t),
\]
and the coefficients will be given by

\[ A_{mn} h'_{mn}(0) = \frac{\iint \beta(r, \theta) f_{mn}(r) \cos(m\theta) \, r \, dr \, d\theta}{\iint f_{mn}(r)^2 (\cos m\theta)^2 r \, dr \, d\theta}, \]

and \( B_{mn} \) is the same with sines instead of cosines.