Math 241

Some solutions for Homework 4

7.9.3a. Separating $u(r, \theta, z, t) = b(r)q(\theta)g(z)h(t)$ gives

\[
\begin{align*}
\rho^2 b'' + \rho b' + (\mu \rho^2 - \nu)b &= 0, & b(a) &= 0, & |b(0)| &< \infty \\
q'' + \nu q &= 0, & q(0) &= q\left(\frac{\pi}{2}\right) = 0 \\
g'' + (\lambda - \mu)g &= 0, & g(0) &= g(H) = 0 \\
h' + \lambda \kappa h &= 0
\end{align*}
\]

Solving for $q$ gives $q(\theta) = \sin(2m\theta)$, with $\nu = (2m)^2$. Then solving for $b$ gives $b(r) = J_{2m}(\sqrt{\mu} r)$; using the boundary condition $b(a) = 0$ gives $\mu_{mn} = \left(\frac{z_{(2m)n}}{a}\right)^2$, where $z_{(2m)n}$ is the $n$th positive zero of $J_{2m}$. Solving for $g$ gives $g(z) = \sin\left(\frac{\nu}{H} z\right)$, with $\lambda - \mu = \left(\frac{\nu}{H}\right)^2$. Then we have $\lambda_{mn} = \left(\frac{z_{(2m)n}}{a}\right)^2 + \left(\frac{\nu}{H}\right)^2$, and $h(t) = e^{-k\lambda_{mn}t}.$

So the general solution is

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{mnp} J_{2m}(\sqrt{\mu_{mn} r}) \sin(2m\theta) \sin\left(\frac{\nu}{H} z\right) e^{-k\lambda_{mn} t}.
\]

The initial conditions give

\[
f(r, \theta, z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{mnp} J_{2m}(\sqrt{\mu_{mn} r}) \sin(2m\theta) \sin\left(\frac{\nu}{H} z\right),
\]

so

\[
A_{mnp} = \frac{\iint f(r, \theta, z) r J_{2m}(\sqrt{\mu_{mn} r}) \sin(2m\theta) \sin\left(\frac{\nu}{H} z\right) dr d\theta dz}{\iint \left( J_{2m}(\sqrt{\mu_{mn} r}) \sin(2m\theta) \sin\left(\frac{\nu}{H} z\right) \right)^2 r dr d\theta dz}.
\]

7.10.2c. Separating $u(\rho, \theta, \phi, t) = w(\rho, \theta, \phi) h(t)$ gives

\[
\begin{align*}
\Delta w + \lambda w &= 0, & w(a, \theta, \phi) &= 0 \\
h' + \kappa \lambda h &= 0.
\end{align*}
\]

We solved for $w$ in class; we get product solutions

\[
w = \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda_{np}} \rho) \begin{pmatrix} \cos m\theta \\ \sin m\theta \end{pmatrix} P_n^m(\cos \phi),
\]

where $n \geq m$. Here both $\cos m\theta$ and $\sin m\theta$ will give us product solutions, and $\lambda_{np} = \left(\frac{z_{(n+\frac{1}{2})p}}{a}\right)^2$, where $z_{(n+\frac{1}{2})p}$ is the $p$th positive zero of $J_{n+\frac{1}{2}}$.

Solving for $h$ gives $h(t) = e^{-k\lambda_{np}t}$.

So

\[
u = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{p=1}^{\infty} \rho^{-1/2} J_{n+\frac{1}{2}}(\sqrt{\lambda_{np}} \rho) (A_{mnp} \cos m\theta + B_{mnp} \sin m\theta) P_n^m(\cos \phi) e^{-k\lambda_{np} t}.
\]
If you want, omit the sin term when \( m = 0 \), but it doesn’t matter because the initial conditions are

\[
u(\rho, \theta, \phi, t) = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \sum_{p=1}^{\infty} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{np}} \rho)(A_{mnp} \cos m\theta + B_{mnp} \sin m\theta) P_n^m(\cos \phi) = F(\rho, \phi) \cos \theta,
\]

so only the \( \cos \theta \) terms will remain anyway. So \( A_{mnp} = 0 \) if \( m \neq 0 \), and \( B_{mnp} = 0 \) always. So our solution becomes

\[
\sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{1np} \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{np}} \rho) \cos(\theta) P_n^1(\cos \phi) e^{-k\lambda_{np}},
\]

and

\[
A_{1np} = \frac{\iint F(\rho, \phi) \rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{np}} \rho) \cos(\theta) P_n^1(\cos \phi) \cos \theta \rho^2 \sin \phi \rho \, d\phi \, d\rho \, d\theta}{\iint (\rho^{-1/2} J_{n+1/2}(\sqrt{\lambda_{np}} \rho) \cos(\theta) P_n^1(\cos \phi))^2 \rho^2 \sin \phi \rho \, d\phi \, d\rho \, d\theta}.
\]

7.10.10b. (I didn’t notice this at the time, but you need to assume that \( u \) is bounded when \( \rho \) goes to \( \infty \), and the problem doesn’t explicitly say so.) We want to solve \( \Delta u = 0 \). Since the boundary conditions are symmetric about the \( z \)-axis (i.e. the axis of the earth), we expect \( u \) to be too. So \( u \) shouldn’t depend on \( \theta \). So we separate

\[
u(\rho, \phi) = f(\rho) g(\phi),
\]

get

\[
\frac{d}{d\rho} \left( \rho^2 f' \right) - \mu f = 0 \quad |f(\infty)| < \infty \quad |f(0)| \leq \infty, |g(\pi)| < \infty
\]

(Here I’m using \( \mu \). Otherwise I would’ve used \( \lambda \).) Note that there are no homogeneous boundary conditions explicitly spelled out in the problem statement; we needed to come up with the boundedness on our own.

Anyway, the solution to the \( \phi \) equation is \( P_n^0(\cos \phi) \) and \( \mu = n(n+1) \), where \( n \geq 0 \). Plugging this into the \( \rho \) equation gives

\[
\frac{d}{d\rho} \left( \rho^2 f' \right) - n(n+1) f = 0.
\]

This is a Cauchy-Euler equation whose solution is \( f(\rho) = c_1 \rho^n + c_2 \rho^{-n-1} \). We don’t want \( f \) to blow up when \( \rho \to \infty \), so we set \( c_1 = 0 \). So our product solutions will be \( P_n^0(\cos \phi) \rho^{-n-1} \), and the general solution is

\[
u(\rho, \phi) = \sum_{n=0}^{\infty} A_n P_n^0(\cos \phi) \rho^{-n-1}.
\]

To find \( A_n \), we use the BCs:

\[
u(a, \phi) = \sum_{n=0}^{\infty} A_n a^{-n-1} P_n^0(\cos \phi) = F(\phi).
\]

So we have

\[
A_n a^{-n-1} = \frac{\int_0^\pi F(\phi) P_n^0(\cos \phi) \sin \phi \, d\phi}{\int_0^\pi P_n^0(\cos \phi)^2 \sin \phi \, d\phi}.
\]
8.3.1c. For a reference temperature, we use \( r(x,t) = A(t) \). Plugging \( u(x,t) = v(x,t) + A(t) \) into the PDE gives
\[
\frac{\partial}{\partial t}(v + A(t)) = k \frac{\partial^2}{\partial x^2}(v + A(t)) + Q(x,t),
\]
which we rearrange to
\[
v_t = kv_{xx} + \bar{Q}(x,t),
\]
where \( \bar{Q} = Q - A'(t) \). For BCs and ICs, we have
\[
\begin{align*}
v(0,t) &= 0 \\
v_x(L,t) &= 0 \\
v(x,0) &= f(x) - A(0).
\end{align*}
\]
The homogeneous problem is \( v_t = kv_{xx} \); the corresponding eigenvalue problem is \( \phi'' + \lambda \phi = 0 \), with \( \phi(0) = \phi'(L) = 0 \). Then we have \( \phi_n = \sin \left( \frac{(2n-1)\pi}{L} x \right) \), and \( \lambda_n = \left( \frac{(2n-1)\pi}{L} \right)^2 \). We expand \( v \) in terms of the eigenfunctions:
\[
v(x,t) = \sum_{n=1}^{\infty} a_n(t) \phi_n(x),
\]
and while we’re at it, let’s expand \( \bar{Q} \):
\[
\bar{Q}(x,t) = \sum_{n=1}^{\infty} q_n(t) \phi_n(x),
\]
where
\[
q_n(t) = \frac{\int_0^L Q(x,t) \phi_n(x) \, dx}{\int_0^L \phi_n(x)^2 \, dx}.
\]
We have
\[
v_t = \sum_{n=1}^{\infty} a_n'(t) \phi_n(x)
\]
and
\[
v_{xx} = \sum_{n=1}^{\infty} a_n(t) \phi_n''(x) = -\sum_{n=1}^{\infty} \lambda_n a_n(t) \phi_n(x).
\]
Expanding the equation \( v_t - kv_{xx} = Q \) gives
\[
\sum_{n=1}^{\infty} (a_n'(t) + k \lambda_n a_n(t)) \phi_n(x) = \sum_{n=1}^{\infty} q_n(t) \phi_n(x).
\]
So \( a_n' + k \lambda_n a_n = q_n \); solving (by integrating factors, for example), gives \( a_n(t) = e^{-k\lambda t}(a_n(0) + \int_0^t e^{k\lambda \tau} q_n(\tau) \, d\tau) \). We solve for \( a_n(0) \) with the initial conditions
\[
v(x,0) = \sum_{n=1}^{\infty} a_n(0) \phi_n(x) = f(x) - A(0),
\]
so
\[ a_n(0) = \frac{\int_0^L (f(x) - A(0))\phi_n(x) \, dx}{\int_0^L \phi_n(x)^2 \, dx}. \]

Then \( u(x, t) = v(x, t) + A(t). \) So,
\[ u(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) + A(t), \]
where \( a_n(t) \) and \( \phi_n(x) \) are given above.

8.3.5. We use \( r(x, t) = 1 - \frac{x}{\pi}. \) Then \( v_t = v_{xx} + e^{-2t} \sin 5x, \) \( v(0, t) = v(\pi, t) = 0, \) and \( v(x, 0) = \frac{x}{\pi} - 1. \) The corresponding eigenvalue problem is \( \phi'' + \lambda \phi = 0, \) \( \phi(0) = \phi(\pi) = 0. \) So we have \( \phi_n(x) = \sin(nx) \) with \( \lambda_n = n^2. \)

Expand \( v \) into eigenfunctions:
\[ v(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \]

and \( Q(x, t) = e^{-2t} \sin 5x: \)
\[ e^{-2t} \sin 5x = \sum_{n=1}^{\infty} q_n(t) \sin(nx). \]

We can see directly that \( q_5(t) = e^{-2t}, \) and \( q_n(t) = 0 \) for \( n \neq 5. \) As in the previous problem, we get
\[ a_n' + \lambda_n a_n = q_n(t). \]

For \( n = 5, \) we have
\[ a_5' + 25a_5(t) = e^{-2t}; \]
we can solve by an integrating factor or the method of undetermined coefficients:
\[ a_5(t) = a_5(0)e^{-25t} + \frac{1}{23}(e^{-2t} - e^{-25t}). \]

For \( n \neq 5, \) we just get
\[ a_n' + n^2a_n = 0, \]
so \( a_n(t) = a_n(0)e^{-n^2t}. \) For initial conditions, we use
\[ v(x, 0) = \sum_{n=1}^{\infty} a_n(0)\sin(nx) = \frac{x}{\pi} - 1, \]
so \( a_n(0) = \frac{2}{\pi} \int_0^\pi \sin(nx)(1 - \frac{x}{\pi}) \, dx = \frac{2}{n\pi}. \) Then,
\[ v(x, t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \]
\[ = \left( \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n^2t} \sin(nx) \right) + \frac{1}{23}(e^{-2t} - e^{-25t}), \]
and
\[ u(x, t) = v(x, t) + 1 - \frac{x}{\pi} \]
\[ = \left( \sum_{n=1}^{\infty} \frac{2}{n\pi} e^{-n^2 t} \sin(nx) \right) + \frac{1}{23} (e^{-2t} - e^{-25t}) + 1 - \frac{x}{\pi}, \]

8.6.1d. The total flux in is 0, and the total heat from sources is \( \iint Q(x, y) \, dx \, dy \). For there to be an equilibrium, we need no change in the total heat, so \( \iint Q(x, y) \, dx \, dy = 0 \).

Let’s expand \( u \) in terms of the eigenfunctions of \( \phi'' + \lambda \phi = 0 \), where \( \phi \) has the same BCs as \( u \). Separating \( \phi(x, y) = f(x)g(y) \) gives
\[ f'' + \mu f = 0 \quad f'(0) = f'(L) = 0 \]
\[ g'' + (\lambda - \mu)g \quad g'(0) = g'(H) = 0. \]

So we have \( f(x) = \cos(\frac{m\pi x}{L}) \) and \( \mu = (\frac{m\pi}{L})^2 \), and \( g(y) = \cos(\frac{n\pi y}{H}) \) and \( \lambda - \mu = (\frac{n\pi}{H})^2 \). Here \( m = 0, 1, 2, \ldots \) and \( n = 0, 1, 2, \ldots \). Then \( \phi_{mn} = \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi y}{H}) \), with \( \lambda_{mn} = (\frac{m\pi}{L})^2 + (\frac{n\pi}{H})^2 \).

Then
\[ u(x, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi y}{H}). \]

Setting \( \Delta u = Q(x, y) \), we have
\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-\lambda_{mn}) A_{mn} \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi y}{H}) = Q(x, y). \]

If we let
\[ Q(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q_{mn} \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi y}{H}), \]
with
\[ q_{mn} = \iint Q(x, y) \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi y}{H}) \, dx \, dy \]
\[ \iint \left( \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi y}{H}) \right)^2 \, dx \, dy, \]
then \( -\lambda_{mn} A_{mn} = q_{mn} \). Note that \( \lambda_{00} = 0 \), so we need \( q_{00} = 0 \). But if either \( m \) or \( n \) is nonzero, then \( \lambda_{mn} \neq 0 \), so we have \( A_{mn} = -\frac{q_{mn}}{\lambda_{mn}} \). Note that \( A_{00} \) can be anything.