Each exercise is worth 50 points.

**Exercise 1.** a) The operator $L_1$ is defined on smooth functions of $(x, y)$ by:

$$L_1(u) := \text{arctan}(xy) \cdot u_{xx} + \sin(x^2y^2) \cdot u_{yy}.$$  

Is the operator $L_1$ linear? Prove your answer.

b) Does the answer change if we replace the operator $L_1$ by the operator $L_2$, which is given by:

$$L_2(u) := u_{xx} + e^u ?$$

c) Find the general solution of the PDE $u_x + x^2 u_y = 0$ by using the method of characteristics. Check that your solution solves the PDE. You don't need to show that these are all of the solutions.

**Solution:**

a) Given smooth functions $u, v$ and constants $a, b$, we compute:

$$L_1(au + bv) = \text{arctan}(xy) \cdot (au + bv)_{xx} + \sin(x^2y^2) \cdot (au + bv)_{yy} =
$$

$$= a\left( \text{arctan}(xy) \cdot u_{xx} + \sin(x^2y^2) \cdot u_{yy} \right) + b\left( \text{arctan}(xy) \cdot u_{xx} + \sin(x^2y^2) \cdot u_{yy} \right) =
$$

$$= aL_1(u) + bL_1(v).$$

Hence, $L_1$ is linear.

b) We note that $L_2(0) = 1 \neq 0$, which implies that the operator is not linear. Namely, for a linear operator $T$, we know that $T(0) = 0$ if we substitute $a = b = 0$ into the definition of linearity.

c) The characteristic ODE is given by:

$$\frac{dy}{dx} = x^2.$$  

The general solution is given by:

$$y(x) = \frac{x^3}{3} + C.$$  

Hence, by using the method of characteristics, the solution $u$ is given by:

$$u(x, y) = f\left( y - \frac{x^3}{3} \right)$$  

for some (differentiable) function $f : \mathbb{R} \to \mathbb{R}$.

For $u$ defined as above, we note that:

$$u_x(x, y) = -x^2f'\left( y - \frac{x^3}{3} \right)$$

and

$$u_y(x, y) = f'\left( y - \frac{x^3}{3} \right).$$

Hence:

$$u_x + x^2u_y = -x^2f'\left( y - \frac{x^3}{3} \right) + x^2f'\left( y - \frac{x^3}{3} \right) = 0. \quad \square$$
Exercise 2. In this exercise, we would like to find a solution to the following initial value problem:

\[ \begin{align*}
    u_t - u_{xx} &= 0, \text{ for } x \in \mathbb{R}, t > 0 \\
    u(x, 0) &= x^2, \text{ for } x \in \mathbb{R}.
\end{align*} \]  

(a) Let \( v := u_{xxx} \). What initial value problem does \( v \) solve?

(b) Use this observation to deduce that we can take \( v = 0 \) to be a solution of the initial value problem obtained in part (a).

c) What does this tell us about the form of \( u \)?

d) Use the latter expression to find a solution of (1). Check that the obtained function solves (1).

e) Alternatively, write the formula for a solution of (1) involving the heat kernel on \( \mathbb{R} \). Write the heat kernel explicitly in terms of exponentials. Don’t simplify the integral.

Solution:

(a) By the differentiation property of the heat equation, we deduce that \( v \) also solves the heat equation. We note that \( v_{xxx}(x, 0) = 0 \). Hence, \( v \) solves the initial value problem:

\[ \begin{align*}
    v_t - v_{xx} &= 0, \text{ for } x \in \mathbb{R}, t > 0 \\
    v(x, 0) &= 0, \text{ for } x \in \mathbb{R}.
\end{align*} \]

(b) We note that the function \( v = 0 \) solves the initial value problem in part a).

c) From part b), we observe that we can look for a solution to (1) of the form:

\[ u(x, t) = A(t) + B(t) \cdot x + C(t) \cdot x^2 \]

for some (differentiable) functions \( A, B, C : \mathbb{R}^+_t \to \mathbb{R} \) satisfying \( A(0) = B(0) = 0, C(0) = 1 \).

d) We note that, for \( u \) of the form (2), one has:

\[ u_t - u_{xx} = (A'(t) - 2C(t)) + B'(t) \cdot x + C'(t) \cdot x^2 \]

Hence, such a \( u \) solves the heat equation if and only if:

\[ \begin{align*}
    A'(t) &= 2C(t) \\
    B'(t) &= 0 \\
    C'(t) &= 0.
\end{align*} \]

From the latter two conditions, it follows that \( B \) and \( C \) are constant. Since \( B(0) = 0 \) and \( C(0) = 1 \), we deduce that:

\[ B(t) = 0 \text{ and } C(t) = 1. \]

We now use the first condition to deduce that:

\[ A'(t) = 2C(t) = 2. \]

Since \( A(0) = 0 \), we conclude that \( A(t) = 2t \). Putting all of this together, we obtain:

\[ u(x, t) = 2t + x^2. \]

We readily check that the function \( u \) defined in (3) solves the initial value problem (1). Namely:

\[ u_t = u_{xx} = 2, \text{ hence } u_t - u_{xx} = 0 \text{ and } u(x, 0) = 0 + x^2 = x^2. \]
e) We use the formula from class and recall that we are taking the diffusion coefficient to equal to 1, and hence $u$ given by:

\[
(4) \quad u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4t}} \cdot y^2 \, dy
\]
solves (1). □

Exercise 3. a) Show that the function $u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $u(x) := \log |x|$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$.

In the following, suppose that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function which equals zero outside of some ball centered at the origin.

b) Prove that:

\[
\lim_{\epsilon \to 0} \int_{\partial B(0,\epsilon)} \left[ \log |x| \cdot \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n} \left( \log |x| \right) \cdot \phi(x) \right] dS(x) \to 2\pi \phi(0).
\]

for $n$ being the unit normal on $\partial B(0,\epsilon)$ pointing towards the origin.

c) Use the result from part b) in order to prove:

\[
\phi(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x| \cdot \Delta \phi(x) \, dx.
\]

Solution:

a) We write $\log |x|$ as $\log \sqrt{x_1^2 + x_2^2}$.

By the Chain Rule, it follows that:

\[
(\log |x|)_{x_1} = \frac{1}{\sqrt{x_1^2 + x_2^2}} \cdot \frac{2x_1}{2\sqrt{x_1^2 + x_2^2}} = \frac{x_1}{x_1^2 + x_2^2}.
\]

\[
(\log |x|)_{x_2} = \frac{1}{x_1^2 + x_2^2} - \frac{2x_1^2}{(x_1^2 + x_2^2)^2}.
\]

By symmetry:

\[
(\log |x|)_{x_2} = \frac{1}{x_1^2 + x_2^2} - \frac{2x_2^2}{(x_1^2 + x_2^2)^2}.
\]

Summing the previous two identities, we obtain:

\[
\Delta \log |x| = 0.
\]

Alternatively, we can use the formula for Laplace's operator in polar coordinates:

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}
\]

in order to deduce that:

\[
\Delta \log r = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \log r = -\frac{1}{r^2} + \frac{1}{r^2} = 0.
\]

b) \[
\lim_{\epsilon \to 0} \int_{\partial B(0,\epsilon)} \left[ \log |x| \cdot \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n} \left( \log |x| \right) \cdot \phi(x) \right] dS(x) \to 2\pi \phi(0).
\]

Let us first observe that, there exists $M > 0$ independent of $\epsilon$ such that, when $\epsilon$ is sufficiently small, it is the case that:

\[
|\frac{\partial \phi}{\partial n}| \leq M.
\]
Consequently:
\[ \left| \int_{\partial B(0, \epsilon)} \log |x| \frac{\partial \phi}{\partial n} \, dS \right| = \left| \log(\epsilon) \right| \left| \int_{\partial B(0, \epsilon)} \frac{\partial \phi}{\partial n} \, dS \right| \leq \left| \log(\epsilon) \right| \int_{\partial B(0, \epsilon)} \left| \frac{\partial \phi}{\partial n} \right| \, dS \leq 2\pi M \epsilon \left| \log(\epsilon) \right|. \]

Let us now observe that:
\[ \lim_{x \to 0^+} (x \log x) = 0. \]

This fact follows from L'Hôpital's rule since:
\[ \lim_{x \to 0^+} \frac{\log x}{x} = \lim_{x \to 0^+} \frac{(\log x)'}{1} = \lim_{x \to 0^+} \frac{1}{x} = \lim_{x \to 0^+} (-x) = 0. \]

Consequently, the integral of the first term goes to zero as \( \epsilon \to 0 \).

We now need to look at the integral of the second term, Let us note that:
\[ \frac{\partial}{\partial n} \log |x| = (\nabla \log |x|) \cdot n. \]

From the calculations in part a), it follows that:
\[ \nabla \log |x| = \left( (\log |x|)_{x_1}, (\log |x|)_{x_2} \right) = \left( \frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2} \right) = \frac{x}{|x|^2}. \]

By definition, we obtain that on \( \partial B(0, \epsilon) \), one has:
\[ n = -\frac{x}{\epsilon}. \]

Hence:
\[ \frac{\partial}{\partial n} \log |x| = -\frac{1}{\epsilon} \frac{x \cdot x}{|x|^2} = -\frac{1}{\epsilon}. \]

Alternatively, we can use polar coordinates and see that:
\[ \frac{\partial}{\partial n} \log |x| = -\frac{\partial}{\partial r} \log r = -\frac{1}{r} = -\frac{1}{\epsilon} \]
on \( \partial B(0, \epsilon) \). It follows that:
\[ \int_{\partial B(0, \epsilon)} \left[ -\frac{\partial}{\partial n} \left( \log |x| \right) \right] \cdot \phi(x) \, dS(x) = \frac{1}{\epsilon} \int_{\partial B(0, \epsilon)} \phi(x) \, dS(x) \to 2\pi \phi(0) \text{ as } \epsilon \to 0. \]

**c)** Let us assume that \( \phi = 0 \) outside of \( B(0, R) \subseteq \mathbb{R}^2 \) and let \( \epsilon \in (0, R) \) be given. We let:
\[ \Omega_\epsilon := B(0, 2R) \setminus B(0, \epsilon). \]

From part a), we know that, on \( \mathbb{R}^2 \setminus \{0\} \):
\[ \Delta \log |x| = 0. \]

We now apply Green's second identity, noting that \( \phi \) and \( \log |x| \) are both smooth on \( \Omega_\epsilon \) in order to deduce that:
\[ \int_{\Omega_\epsilon} \left[ \log |x| \cdot \Delta \phi(x) - \Delta \log |x| \cdot \phi(x) \right] \, dx = \int_{\partial \Omega_\epsilon} \left[ \log |x| \cdot \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n} \left( \log |x| \right) \cdot \phi(x) \right] \, dS(x). \]

We note that \( \partial \Omega_\epsilon \) consists of two parts: \( \partial B(0, \epsilon) \) and \( \partial B(0, 2R) \). Since, by assumption, \( \phi \) vanishes near \( \partial B(0, 2R) \), it follows that the contribution to the right-hand side from the outer boundary \( \partial B(0, 2R) \) equals to zero. Moreover, we know that \( \Delta \log |x| = 0 \) on \( \Omega_\epsilon \). Hence, it follows that:
\[ \int_{\Omega_\epsilon} \log |x| \cdot \Delta \phi(x) \, dx = \int_{\partial B(0, \epsilon)} \left[ \log |x| \cdot \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n} \left( \log |x| \right) \cdot \phi(x) \right] \, dS(x). \]

We note that \( \Delta \phi = 0 \) for \( |x| \geq 2R \) and we deduce that:
\[ \int_{|x| \geq \epsilon} \log |x| \cdot \Delta \phi(x) \, dx = \int_{\partial B(0, \epsilon)} \left[ \log |x| \cdot \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n} \left( \log |x| \right) \cdot \phi(x) \right] \, dS(x). \]
We now let $\epsilon \to 0$ and we use the result from part b) in order to deduce that:

$$\int_{\mathbb{R}^2} \log |x| \cdot \Delta \phi(x) \, dx = 2\pi \phi(0).$$

The claim now follows. □

**Exercise 4.** Let us recall the representation formula for harmonic functions in three dimensions:

For $\Omega \subseteq \mathbb{R}^3$ a bounded domain, $u$ a harmonic function on $\Omega$ which extends continuously up to $\partial \Omega$, and $x_0 \in \Omega$, the following formula holds:

$$u(x_0) = \frac{1}{4\pi} \int_{\partial \Omega} \left[ -u(y) \frac{\partial}{\partial n} \left( \frac{1}{|y-x_0|} \right) + \frac{1}{|y-x_0|} \frac{\partial u}{\partial n} \right] \, dS(y).$$

Here, $n$ denotes the outward pointing unit normal on $\partial \Omega$.

In this exercise, one is allowed to use the representation formula without proof.

a) State the mean value property for harmonic functions in three-dimensions.

b) Use the representation formula in order to prove the mean value property in three dimensions.

c) State the definition of the Green’s function $G(x, x_0)$ for the Laplace operator on a three-dimensional domain $\Omega$ with $x_0$ a point in $\Omega$.

d) Use the representation formula and properties of the Green’s function to show that the harmonic function $u$ defined in the beginning of the problem satisfies:

$$u(x_0) = \int_{\partial \Omega} u(y) \cdot \frac{\partial G(x, x_0)}{\partial n} \, dS(y).$$

**Solution:**

a) Let $x_0 \in \mathbb{R}^3$ and $R > 0$ are given and suppose that $u : B(x_0, R) \to \mathbb{R}^3 \to \mathbb{R}$ is a harmonic function which extends continuously up to $\partial B(x_0, R)$. The mean value property then states that:

$$u(x_0) = \frac{1}{4\pi R^2} \int_{\partial B(x_0, R)} u(y) \, dS(y).$$

b) We can replace the function $u$ with the function $v(x) := u(x - x_0)$ to see that it suffices to prove the claim in the special case when $x = 0$. It is important to note that the function $v$ is harmonic if the function $u$ is harmonic.

In other words, we are assuming that $u : B(0, R) \to \mathbb{R}$ is harmonic and that it extends continuously up to $\partial B(0, R)$ and we want to prove that:

$$u(0) = \frac{1}{4\pi R^2} \int_{\partial B(0, R)} u(y) \, dS(y)$$

by using the fact that:

$$u(0) = \frac{1}{4\pi} \int_{\partial B(0, R)} \left[ -u(y) \frac{\partial}{\partial n} \left( \frac{1}{|y|} \right) + \frac{1}{|y|} \frac{\partial u}{\partial n} \right] \, dS(y).$$

Let us first note that we can use polar coordinates to deduce that, on $\partial B(0, R)$, we can write $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$. Hence:

$$\frac{\partial}{\partial n} \left( \frac{1}{|y|} \right) = \frac{\partial}{\partial r} \left( \frac{1}{r} \right) = -\frac{1}{r^2}. $$
It follows that:

\[
\frac{1}{4\pi} \int_{\partial B(0,R)} \left[ -u(y) \frac{\partial}{\partial n} \left( \frac{1}{|y|} \right) \right] dS(y) = \frac{1}{4\pi R^2} \int_{\partial B(0,R)} u(y) dS(y). \tag{5}
\]

Moreover, we note that:

\[
\frac{1}{4\pi} \int_{\partial B(0,R)} \frac{1}{|y|} \frac{\partial u}{\partial n} dS(y) = \frac{1}{4\pi R} \int_{\partial B(0,R)} \nabla u \cdot n dS(y)
\]

which, by the Divergence Theorem equals:

\[
\frac{1}{4\pi R} \int_{B(0,R)} \Delta u(y) dy = 0. \tag{6}
\]

The claim now follows from (5) and (6).

c) Suppose that \( \Omega \subseteq \mathbb{R}^3 \) is a bounded domain with \( x_0 \in \Omega \). The Green’s function \( G(x, x_0) \) is a function defined on \( \Omega \setminus \{x_0\} \), which is continuous up to \( \partial \Omega \), and which satisfies the following properties:

1) \( G(x, x_0) \) is a harmonic function on \( \Omega \setminus \{x_0\} \).
2) \( G(x, x_0) = 0 \) for \( x \in \partial \Omega \).
3) \( H(x, x_0) := G(x, x_0) + \frac{1}{4\pi|x-x_0|} \) is harmonic on \( \Omega \).

d) We recall from property 3) in part c) that the function \( H(x, x_0) = G(x, x_0) + \frac{1}{4\pi|x-x_0|} \) is harmonic on \( \Omega \). We also know that the function \( u \) is harmonic on \( \Omega \). Hence, by Green’s second identity:

\[
0 = \int_{\partial \Omega} \left[ u(x) \frac{\partial H(x, x_0)}{\partial n} - H(x, x_0) \frac{\partial u}{\partial n} \right] dS(x).
\]

By the representation formula, we know that:

\[
u(x_0) = \int_{\partial \Omega} \left[ u(x) \frac{\partial G(x, x_0)}{\partial n} - G(x, x_0) \frac{\partial u}{\partial n} \right] dS(x).
\]

By using property 3) of Green’s functions, it follows that:

\[
u(x_0) = \int_{\partial \Omega} \left[ u(x) \frac{\partial G(x, x_0)}{\partial n} - G(x, x_0) \frac{\partial u}{\partial n} \right] dS(x)
\]

By property 2), we know that \( G(x, x_0) = 0 \) for \( x \in \partial \Omega \). Hence:

\[
u(x_0) = \int_{\partial \Omega} u(x) \frac{\partial G(x, x_0)}{\partial n} dS(x)
\]

as was claimed. □

**Exercise 5.** Throughout this exercise, we assume that \( c > 0 \) is a constant.

a) Consider the differential operator \( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \), defined on smooth functions of \( (x, t) \in \mathbb{R} \times \mathbb{R} \).

Show that there exist first-order differential operators \( T_1 \) and \( T_2 \) such that for all smooth functions \( u : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \), the following identity holds:

\[
\left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u = T_1 T_2 u.
\]

b) What is the physical interpretation of the operators \( T_1 \) and \( T_2 \)?

c) Using the above factorization, show that the general solution to the wave equation on \( \mathbb{R} \times \mathbb{R} \):

\[
u_{tt} - c^2 u_{xx} = 0
\]

is given by:

\[
u(x, t) = f(x - ct) + g(x + ct)
\]
for some functions $f, g : \mathbb{R} \to \mathbb{R}$.

d) Check that the function $u$ obtained in part c) solves the wave equation. How many derivatives do the functions $f$ and $g$ need to have in order for this calculation to be rigorous?

**Solution:**

a) We note that:

$$\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} = \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right).$$

More precisely, for $u : \mathbb{R} \times \mathbb{R}_t \to \mathbb{R}$, the following identity holds:

$$\left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = \left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( u_t + cu_x \right) =
\begin{align*}
= & \ u_{tt} + cu_{xt} - cu_{tx} - c^2 u_{xx} = u_{tt} - c^2 u_{xx} = \left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u.
\end{align*}$$

Hence, we can take:

$$T_1 = \frac{\partial}{\partial t} - c \frac{\partial}{\partial x},$$
$$T_2 = \frac{\partial}{\partial t} + c \frac{\partial}{\partial x}.$$ 

b) The operators $T_1$ and $T_2$ correspond to transport with speed $c$ to the left and to the right respectively.

c) Suppose that:

$$u_{tt} - c^2 u_{xx} = 0.$$ 

By part a), we can write this equation as:

$$\left( \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) u = 0.$$ 

Let us take:

$$G(x, t) := u_t + cu_x.$$ 

We can then deduce that:

$$G_t - cG_x = 0.$$ 

Hence, $G$ solves the transport equation. It follows that:

$$G(x, t) = H(x + ct)$$ 

for some (differentiable) function $H : \mathbb{R} \to \mathbb{R}$. We substitute this back into the definition of the function $G$ to deduce that $u$ then has to solve:

$$u_t + cu_x = G(x, t) = H(x + ct).$$

In particular, $u$ solves an inhomogeneous transport equation. We note that the general solution of the associated homogeneous equation:

$$u^{(h)}_{tt} + cu^{(h)}_{xx} = 0$$

is given by:

$$u^{(h)}(x, t) = f(x - ct)$$

for some (differentiable) function $f : \mathbb{R} \to \mathbb{R}$. Hence, we need to find a particular solution $u^{(p)}$ of (7). Since the right-hand side is a function of $x + ct$, we look for a particular solution which is a function of $x + ct$ as well. In particular, we look for a solution of the form:

$$u^{(p)}(x, t) = g(x + ct)$$
for some (differentiable) function $g : \mathbb{R} \to \mathbb{R}$. For $u^{(p)}$ defined as above, we note that:

$$u_t^{(p)} + cu_x^{(p)} = (1 + c) \cdot g'(x + ct).$$

Hence, we want to choose $h$ in such a way that:

$$(1 + c)g'(x + ct) = H(x + ct).$$

In particular, we can take:

$$g(y) := \frac{1}{1+c} \int_0^y H(s) \, ds.$$  

Consequently, we obtain that:

$$u(x, t) = f(x - ct) + g(x + ct).$$

d) For $u$ defined as in part c), we note that:

$$u_t = -cf'(x - ct) + cg'(x + ct)$$

$$u_{tt} = c^2 f''(x - ct) + c^2 g''(x + ct)$$

$$u_x = f'(x - ct) + g'(x + ct)$$

$$u_{xx} = f''(x - ct) + g''(x + ct).$$

In particular, it follows that:

$$u_{tt} = c^2 u_{xx} = c^2 f''(x - ct) + c^2 g''(x + ct)$$

and so:

$$u_{tt} - c^2 u_{xx} = 0.$$  

In order to make this calculation rigorous, we need to assume that the functions $f, g : \mathbb{R} \to \mathbb{R}$ are twice differentiable. □