MATH 425, HOMEWORK 4

This homework is due on Friday, February 22 (one should leave it in my mailbox by noon). There are three problems. Each problem is worth 10 points.

Exercise 1. Consider the function \( u(x,t) = 1 - x^2 - 2kt \), for \( k > 0 \) a constant.

a) Verify that \( u \) is a solution to the heat equation.

b) Find the minimum and maximum of \( u \) on the closed rectangle \( \{(x,t); 0 \leq x \leq 1, 0 \leq t \leq T\} = [0,1]_x \times [0,T]_t \), for a fixed \( T > 0 \), without using the maximum principle.

c) Find the minimum and maximum of \( u \) on \( [0,1]_x \times [0,T]_t \) by using the maximum principle.

Exercise 2. Consider the initial value problem:

\[
\begin{cases}
    u_t - u_{xx} = 0, & \text{for } 0 < x < 1, \ t > 0 \\
    u(x,0) = 4x(1-x), & \text{for } 0 \leq x \leq 1 \\
    u(0,t) = 0, u(1,t) = 0, & \text{for } t > 0.
\end{cases}
\]

Show that:

\[ u(x,t) = u(1-x,t), \]

for all \( 0 \leq x \leq 1, t \geq 0. \]

[HINT: What boundary value problem does the function \( v(x,t) := u(1-x,t) \) solve?]

Exercise 3. (A comparison principle)

a) Suppose that \( u \) and \( v \) both solve the heat equation on \( [0,L]_x \times (0, +\infty)_t \). Furthermore, suppose that \( u \leq v \) for \( t = 0 \), for \( x = 0 \), and for \( x = L \). Show that:

\[ u \leq v \] on \( [0,L]_x \times [0, +\infty)_t \).

[HINT: Be careful to use the maximum principle on bounded rectangles.]

b) More generally, consider functions \( u \) and \( v \) which solve \( u_t - ku_{xx} = f(x,t) \) and \( v_t - kv_{xx} = g(x,t) \) on \( [0, L]_x \times (0, +\infty)_t \). We assume moreover that \( f \leq g \) on \( [0, L]_x \times (0, +\infty)_t \) and that \( u \leq v \) for \( t = 0 \), for \( x = 0 \), as well as for \( x = L \). Show that \( u \leq v \) on \( [0, L]_x \times [0, +\infty)_t \).

c) Suppose that the function \( v \) satisfies the inequality \( v_t - v_{xx} \geq \sin x \) on \( [0, \pi]_x \times (0, +\infty) \). Moreover, assume that: \( v(0,t) \geq 0, v(\pi,t) \geq 0 \) for all \( t > 0 \) and \( v(x,0) \geq \sin x \) for all \( 0 \leq x \leq \pi \). Show that:

\[ v(x,t) \geq (1 - e^{-t}) \sin x \]
on \( [0, \pi]_x \times [0, +\infty)_t \).