Exercise 1. (Uniqueness for the heat equation on $\mathbb{R}$)

Suppose that the functions $u_1, u_2 : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ solve:

\[
\begin{cases}
\partial_t u_1 - k \cdot \partial^2_x u_1 = 0, \quad x \in \mathbb{R}, \quad t > 0 \\
u_1(x, 0) = \phi(x), \quad x \in \mathbb{R}
\end{cases}
\]

and

\[
\begin{cases}
\partial_t u_2 - k \cdot \partial^2_x u_2 = 0, \quad x \in \mathbb{R}, \quad t > 0 \\
u_2(x, 0) = \phi(x), \quad x \in \mathbb{R}
\end{cases}
\]

for some function $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

Suppose furthermore that there exists constants $C, A > 0$ such that for all $x \in \mathbb{R}$ and for all $t > 0$, one has:

$|u_1(x, t)| \leq Ce^{Ax^2}$ and $|u_2(x, t)| \leq Ce^{Ax^2}$.

Using the Global Maximum Principle (which was stated in class), show that:

$u_1 = u_2$.

(Here, one is allowed to use the result of the Global Maximum Principle, even though we didn’t give the details of its proof in class.)

This type of result is called Conditional Uniqueness. In other words, we know that solutions are unique in the class of objects satisfying some additional condition, which in this case is a bound of the type $|u(x, t)| \leq Ce^{Ax^2}$.

Exercise 2. (The Global Maximum Principle in a special case)

In this Exercise, we will give a proof of a special case of the Global Maximum Principle.

Suppose that

\[
\begin{cases}
\partial_t u - ku_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0 \\
u(x, 0) = \phi(x), \quad x \in \mathbb{R}
\end{cases}
\]

for some bounded continuous function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ which equals zero outside of $[0, 1]$.

Suppose moreover that the solution $u$ is bounded from above, i.e. that there exists $C > 0$ such that:

$u(x, t) \leq C$ for all $x \in \mathbb{R}, \ t > 0$.

Let $M_0$ denote the maximum of $\phi$ (which exists by the assumptions on $\phi$).

We want to prove that:

(1) $u(x, t) \leq M_0$ for all $x \in \mathbb{R}, \ t > 0$.

a) Fix $T > 0, L > 0$ and consider the rectangle: $Q_{L,T} := [-L, L]_x \times [0, T]_t$. Define on $Q_{T,L}$ the function:

$w(x, t) := \frac{2C}{L^2} \left( \frac{x^2}{2} + kt \right) + M_0$.
Check that:

\[ w_t - kw_{xx} = 0. \]

b) Explain how we can deduce that:

\[ w \geq u \] on \( Q_{L,T} \).

[HINT: Recall the comparison results from the previous homework assignment.]

c) Fix \((x_0, t_0)\). By using the result from part b), and by letting \( L \) tend to infinity, deduce the bound (1).

**Exercise 3.** (Separation of variables for an inhomogeneous PDE)

a) Solve the following boundary value problem by using the method of separation of variables:

\[
\begin{aligned}
&u_t - u_{xx} = \sin(2\pi x) + \sin(3\pi x), \quad 0 < x < 1, \ t > 0 \\
u(x, 0) = 0, \quad 0 \leq x \leq 1 \\
u(0, t) = u(1, t) = 0, \quad t > 0.
\end{aligned}
\]

b) Solve the more general problem:

\[
\begin{aligned}
v_t - v_{xx} = m \cdot v + \sin(2\pi x) + \sin(3\pi x), \quad 0 < x < 1, \ t > 0 \\
v(x, 0) = 0, \quad 0 \leq x \leq 1 \\
v(0, t) = v(1, t) = 0, \quad t > 0
\end{aligned}
\]

for \( m \in \mathbb{R} \) a constant.

**Full credit will be given for solving part b) when** \( m \neq 4\pi^2 \) and \( m \neq 9\pi^2 \). **Extra credit will be given for the solution of the problem in the cases** \( m = 4\pi^2 \) and \( m = 9\pi^2 \).