MATH 425, HOMEWORK 6

This homework is due by noon on Friday, March 22. Please leave your assignment in my mailbox. There are three problems. Each problem is worth 10 points.

**Exercise 1.** (Uniqueness for the Poisson equation by using the energy method) 
Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. We assume that for all $f : \Omega \to \mathbb{R}$ and for all $g : \partial \Omega \to \mathbb{R}$, the boundary value problem:

\[
\begin{cases}
\Delta u = f \text{ on } \Omega \\
u = g \text{ on } \partial \Omega
\end{cases}
\]

admits a solution. 
By using the energy method, show that this solution is uniquely determined if we are given $f$ and $g$. 
[HINT: Suppose that $u_1, u_2$ are two solutions. Look at their difference $w := u_1 - u_2$ and find the problem which problem $w$ solves. Multiply the equation for $w$ by $w$ and integrate over $\Omega$. It is helpful to recall Green’s Identities from multivariable calculus.]

**Exercise 2.** (A necessary condition for existence of solutions) 
Suppose that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain and suppose that $f : \Omega \to \mathbb{R}$ and $g : \partial \Omega \to \mathbb{R}$. Consider the boundary value problem:

\[
\begin{cases}
\Delta u = f \text{ on } \Omega \\
\frac{\partial u}{\partial n} = g \text{ on } \partial \Omega.
\end{cases}
\]

Show that the above boundary value problem doesn’t have a solution unless:

\[
\int_\Omega f \, dx \, dy \, dz = \int_{\partial \Omega} g \, dS
\]

Here, we recall that $\frac{\partial u}{\partial n} := \nabla u \cdot \hat{n}$, where $\hat{n}$ is the outward pointing unit normal vector to $\partial \Omega$. [HINT: Integrate the equation over $\Omega$.]

**Exercise 3.** (Subharmonic functions) 
We say that a function $u = u(x)$ is subharmonic if $\Delta u \geq 0$. In particular, every harmonic function is subharmonic. In this exercise, we will study the maximum principle for subharmonic functions.

a) Suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain and suppose that $u$ is a subharmonic function on $\Omega$. Furthermore, assume that $u$ extends to a continuous function on $\Omega = \Omega \cup \partial \Omega$. 
Show that $u$ achieves its maximum value on $\partial \Omega$. In other words:

\[
\max_{\Omega} u = \max_{\partial \Omega} u.
\]

b) Fix $n = 2$ and look at the function $u(x_1, x_2) = x_1^2 + x_2^2$ on the closed unit ball $B(0, 1) = \{(x_1, x_2) \in \mathbb{R}^2, \, x_1^2 + x_2^2 \leq 1\}$. 
Calculate $\Delta u$ and deduce that $u$ is subharmonic.

c) Check that the maximum principle holds for the function $u$ defined in part b) when the domain $\Omega$ is the open unit ball: $\{(x_1, x_2) \in \mathbb{R}^2, \, x_1^2 + x_2^2 < 1\}$.

d) For the function $u$ defined in part b), find where it achieves its minimum on $B(0, 1)$. Is this minimum achieved on the boundary?