MATH 425, MIDTERM EXAM 2, SOLUTIONS.

Each exercise is worth 25 points.

**Exercise 1.** Consider the initial value problem:

\[
\begin{align*}
    u_t - u_{xx} &= 0, \quad 0 < x < 1, \ t > 0 \\
    u(x, 0) &= x(1 - x), \quad 0 \leq x \leq 1 \\
    u(0, t) &= 0, \ u(1, t) = 0, \ for \ t > 0.
\end{align*}
\]  

(a) Find the maximum of the function \( u \) on \([0, 1]_x \times [0, +\infty)_t\).

(b) Show that, for all \(0 \leq x \leq 1, t \geq 0:\)

\[ u(x, t) \geq 0. \]

(c) Show that, for all \(0 \leq x \leq 1, t \geq 0:\)

\[ u(x, t) \leq x(1 - x)e^{-8t}. \]

(d) Given \(x \in [0, 1]\), calculate \(\lim_{t \to \infty} u(x, t)\).

**Solution:**

(a) We observe that the function \( u \) equals zero on the lateral sides \( x = 0 \) and \( x = 1 \). Hence, by the Maximum Principle, it has to achieve its maximum on the bottom side \( t = 0 \). The function \( x(1 - x) \) achieves its maximum \( \frac{1}{4} \) at \( x = \frac{1}{2} \). Hence, the maximum of \( u \) equals \( \frac{1}{4} \) and it is achieved at the point \((x, t) = (\frac{1}{2}, 0)\).

(b) **First solution:** We apply the Minimum Principle. We note by (2) that the function \( u \) is non-negative on the lateral sides \((x = 0 \) and \( x = 1 \)) and on the bottom side \((t = 0)\) of the infinite rectangle \([0, 1]_x \times [0, +\infty)_t\). The claim then follows from the minimum principle. Strictly speaking, we should apply the Minimum Principle stated in class on a finite rectangle \([0, 1]_x \times [0, T)_t\) and then let \( T \to +\infty \).

Second solution: We can apply the Comparison Principle, which was proved in Exercise 3 of Homework Assignment 4. We can summarize this principle as follows:

Suppose that:

\[
\begin{align*}
    v_t - v_{xx} &\geq w_t - w_{xx}, \ for \ 0 < x < 1, \ t > 0 \\
    v(x, 0) &\geq w(x, 0), \ for \ 0 \leq x \leq 1 \\
    v(0, t) &\geq w(0, t), \ v(1, t) \geq w(1, t), \ for \ t > 0.
\end{align*}
\]  

Then:

\[ v(x, t) \geq w(x, t) \]

for all \(x \in [0, 1], t > 0\). In other words, if \( v_t - v_{xx} \geq w_t - w_{xx} \) and if \( v \geq w \) on the bottom and lateral sides of \([0, 1]_x \times [0, +\infty)_t\), then we can deduce that \( v \geq w \) on all of \([0, 1]_x \times [0, +\infty)_t\).

We now apply the Comparison Principle. Let us note \( u = 0 \) on the lateral sides and since \( u \) equals \( x(1 - x) \), which is non-negative, on the bottom side. Hence, we can apply the Comparison Principle with \( v = u \) and with \( w = 0 \) in order to deduce the claim.
c) In part c), we will have to apply the Comparison Principle. Let us take:

\[ v(x, t) := x(1 - x)e^{-8t}. \]

We compute:

\[ v_t(x, t) = -8x(1 - x)e^{-8t} \]
and

\[ v_{xx}(x, t) = -2e^{-8t}. \]

Hence:

\[ v_t(x, t) - v_{xx}(x, t) = -8x(1 - x)e^{-8t} + 2e^{-8t} = 2(1 - 4x(1 - x))e^{-8t}. \]

Let us recall that we are considering \( x \in [0, 1] \) and so:

\[ 1 - 4x(1 - x) \geq 1 - 4 \cdot \frac{1}{4} = 0, \]

since \( x \mapsto x(1 - x) \) achieves its maximum on \([0, 1]\) at the point \( x = \frac{1}{2} \). Hence:

\[ v_t - v_{xx} \geq 0. \]

Let us also note that:

\[ v(x, 0) = u(x, 0) = x(1 - x) \]

for all \( x \in [0, 1] \).

Moreover,

\[ v(0, t) = v(1, t) = u(0, t) = u(1, t) = 0 \]

for all \( t > 0 \). It follows that we can apply the Comparison Principle with \( v = x(1 - x)e^{-8t} \) as above and with \( w = u \), the solution to (2) in order to deduce the claim.

Let us fix \( x \in [0, 1] \). From parts b) and c), it follows that, for all \( t > 0 \):

\[ 0 \leq u(x, t) \leq x(1 - x)e^{-8t}. \]

It follows that the limit as \( t \to \infty \) of \( u(x, t) \) equals zero. \( \square \)

**Exercise 2.** a) Find a solution to the following boundary value problem by separation of variables:

\[
\begin{aligned}
&u_t(x, t) - u_{xx}(x, t) = \sin(5\pi x), \text{ for } 0 < x < 1, t > 0 \\
&u(x, 0) = 0, \text{ for } 0 \leq x \leq 1 \\
&u(0, t) = u(1, t) = 0, \text{ for } t > 0.
\end{aligned}
\]

b) Is this the only solution to (3)?

**Solution:**

a) We look for a solution of the form:

\[ u(x, t) = A(t) \cdot \sin(5\pi x). \]

The reason why we look for such a solution is that the right-hand side of the equation contains a \( \sin(5\pi x) \) term. We expect that this is the only frequency that will be present in the solution. In the form of \( u \) that we are looking for, for each fixed \( t \), the function \( u(x, t) \) has a Fourier sine expansion in terms of \( \sin(5\pi x) \). The coefficient will be a function of \( t \).

Let us note that, for \( u \) defined as in (4), the boundary conditions \( u(0, t) = u(1, t) = 0 \) are satisfied since \( \sin(0) = \sin(5\pi) = 0 \).

Our goal is to choose \( A(t) \) such that \( u \) solves the inhomogeneous heat equation. We compute:

\[ u_t - u_{xx} = \left\{ A'(t) + 25\pi^2 A(t) \right\} \cdot \sin(5\pi x) \]
which, by the equation, equals: 
\[ \sin(5\pi x) \].
We can now equate the coefficient of \( \sin(5\pi x) \) to deduce:
\[ A'(t) + 25\pi^2 A(t) = 1. \]  
(5)
Hence, the condition (5) guarantees that the function \( u \) defined in (4) solves the PDE.
We now need to solve for \( A(t) \). By the condition that \( u(x,0) = A(0) \cdot \sin(5\pi x) \), it follows that \( A(0) = 0 \). Hence, we need to solve the following initial value problem to determine \( A(t) \):
\[
\begin{cases}
A'(t) + 25\pi^2 A(t) = 1 \\
A(0) = 0.
\end{cases}
\]
We solve the ODE by multiplying with the integrating factor \( e^{25\pi^2 t} \). The ODE then becomes:
\[
e^{25\pi^2 t} A'(t) + 25\pi^2 e^{25\pi^2 t} A(t) = e^{25\pi^2 t}
\]
i.e.
\[
(e^{25\pi^2 t} A(t))' = e^{25\pi^2 t}.
\]
Hence:
\[
e^{25\pi^2 t} A(t) = A_0 + \frac{1}{25\pi^2} e^{25\pi^2 t}.
\]
We note that \( A(0) = 0 \) implies that \( A_0 = -\frac{1}{25\pi^2} \). Consequently:
\[
A(t) = \frac{1}{25\pi^2} \left\{ 1 - e^{-25\pi^2 t} \right\}.
\]
It follows that:
\[
u(x,t) = \frac{1}{25\pi^2} \left\{ 1 - e^{-25\pi^2 t} \right\} \cdot \sin(5\pi x).
\]
b) We know from class that the boundary value problem for the heat equation on a spatial interval of finite length admits unique solutions, either by applying the Maximum Principle or by applying the Energy Method. Hence, the function \( u \) from part a) is the unique solution to (3). □

Exercise 3. Let us recall that a function \( u : \mathbb{R}^n \to \mathbb{R} \) is called subharmonic if \( \Delta u \geq 0 \). In particular, every harmonic function is subharmonic.

a) Given a harmonic function \( u : \mathbb{R}^n \to \mathbb{R} \), show that the function \( v := u^2 \) is subharmonic on \( \mathbb{R}^n \).
b) Under which conditions on \( u \) can we deduce that the function \( v \) defined above is harmonic?

Solution:

a) We compute, for \( 1 \leq j \leq n \):
\[
v_{x_j} = (u^2)_{x_j} = 2uu_{x_j}
\]
and so:
\[
v_{x_jx_j} = (u^2)_{x_jx_j} = 2u_{x_j}u_{x_j} + 2u_{x_jx_j}x_j = 2u_{x_j}^2 + 2uu_{x_jx_j},
\]
We sum in \( j = 1, \ldots, n \) in order to deduce:
\[
\Delta v = 2 \sum_{j=1}^n u_{x_j}^2 + 2u \Delta u = 2|\nabla u|^2 + 2u \Delta u.
\]
Since \( \Delta u = 0 \), this quantity equals: \( 2|\nabla u|^2 \) which is non-negative. Hence, \( v \) is subharmonic.

b) From part a), we recall that:
\[
\Delta v = 2|\nabla u|^2.
\]
In particular \( v \) is harmonic if and only if \( \nabla u = 0 \), which is the case if and only if \( u \) is constant. □
Exercise 4. Suppose that \( u : B(0,1) \to \mathbb{R} \) is a harmonic function on the open ball \( B(0,1) \subseteq \mathbb{R}^2 \), which extends to a continuous function on its closure \( \overline{B}(0,1) \).

Suppose that, in polar coordinates:

\[
    u(1, \theta) = 2 + 3 \sin \theta
\]

for all \( \theta \in [0, 2\pi] \).

a) Find the minimum and the maximum of \( u \) on \( \overline{B}(0,1) \).

b) Find the value of \( u \) at the origin.

c) Find an expression for the value of \( u \) at the point \( \left( \frac{1}{2}, \frac{\pi}{2} \right) \) in polar coordinates by using Poisson’s formula. Don’t explicitly evaluate the integral.

d) Does there exist a point in \( B(0,1) \) at which \( u \) takes the value 5?

Solution:

a) We use the Weak Maximum Principle for the Laplace equation in order to deduce that \( u \) achieves its maximum and minimum on the boundary. More precisely:

\[
    \min_{\overline{B}(0,1)} u = \min_{\partial B(0,1)} u
\]

and

\[
    \max_{\overline{B}(0,1)} u = \max_{\partial B(0,1)} u.
\]

We know that for all \( \theta \in [0, 2\pi] \):

\[
    -1 \leq 2 + 3 \sin \theta \leq 5.
\]

Moreover:

\[
    2 + 3 \sin \left( \frac{3\pi}{2} \right) = -1
\]

and

\[
    2 + 3 \sin \left( \frac{\pi}{2} \right) = 5.
\]

Hence:

\[
    \min_{\overline{B}(0,1)} u = \min_{\partial B(0,1)} u = -1
\]

and

\[
    \max_{\overline{B}(0,1)} u = \max_{\partial B(0,1)} u = 5.
\]

b) We can use the Mean Value Property to deduce that the value of \( u \) at the origin equals the average of the function \( u \) on the circle \( \partial B(0,1) \). In particular:

\[
    u(0) = \frac{1}{2\pi} \int_0^{2\pi} (2 + 3 \sin \theta) \, d\theta = 2,
\]

since \( \int_0^{2\pi} \sin \theta \, d\theta = 0 \).

c) We use Poisson’s formula and we compute:

\[
    u \left( \frac{1}{2}, \frac{\pi}{2} \right) = \frac{1 - (\frac{1}{2})^2}{2\pi} \cdot \int_0^{2\pi} \frac{2 + 3 \sin \phi}{(\frac{1}{2})^2 - 2 \cdot \frac{1}{2} \cdot 1 \cos(\frac{\pi}{2} - \phi) + 1} \, d\phi = \frac{3}{2\pi} \cdot \int_0^{2\pi} \frac{2 + 3 \sin \phi}{5 - 4 \sin \phi} \, d\phi.
\]
d) Suppose that there were a point $x_0 \in B(0, 1)$ at which $u(x_0) = 5$, then by part a), it would follow that:

$$u(x_0) = \max_{B(0,1)} u.$$ 

Hence, $u$ achieves its maximum at an interior point. The Strong Maximum Principle would then imply that $u$ was constant. However, $u$ is not constant on the boundary $\partial B(0, 1)$, which gives us a contradiction. Hence, there is no such point $x_0$ in the interior. □