Exercise 1. Evaluate the integral:
\[ \int_{0}^{2\pi} e^{\theta} \sin \theta \, d\theta \]

a) by using Integration by parts.
b) by using complex numbers.

Solution:

a) Method 1: using integration by parts

\[ \int_{0}^{2\pi} e^{\theta} \sin \theta \, d\theta = \left. e^{\theta} \sin \theta \right|_{\theta=0}^{\theta=2\pi} - \int_{0}^{2\pi} e^{\theta} \cos \theta \, d\theta \]

\[ = -e^{\theta} \cos \theta \bigg|_{\theta=0}^{\theta=2\pi} + \int_{0}^{2\pi} e^{\theta} \cos \theta \, d\theta = (-e^{2\pi} + 1) + \int_{0}^{2\pi} e^{\theta} \cos \theta \, d\theta \]

\[ = (-e^{2\pi} + 1) + e^{\theta} \sin \theta \bigg|_{\theta=0}^{\theta=2\pi} - \int_{0}^{2\pi} e^{\theta} \sin \theta \, d\theta = (-e^{2\pi} + 1) - \int_{0}^{2\pi} e^{\theta} \sin \theta \, d\theta \]

Hence,
\[ 2 \int_{0}^{2\pi} e^{\theta} \sin \theta \, d\theta = (-e^{2\pi} + 1) \]
from where we deduce that the value of the wanted integral is:
\[ \frac{-e^{2\pi} + 1}{2} \]

b) Method 2: using Complex numbers

We note that:
\[ \int_{0}^{2\pi} e^{\theta} \sin \theta \, d\theta = \text{Im} \left( \int_{0}^{2\pi} e^{\theta} \cos \theta + i \sin \theta \, d\theta \right) = \text{Im} \left( \int_{0}^{2\pi} e^{\theta} e^{i\theta} \, d\theta \right) \]

\[ = \text{Im} \left( \int_{0}^{2\pi} e^{(1+i)\theta} \, d\theta \right) = \text{Im} \left( \frac{1}{1+i} e^{(1+i)\theta} \bigg|_{\theta=0}^{\theta=2\pi} \right) = \text{Im} \left( \frac{1}{1+i} (e^{2\pi} - 1) \right) \]

\[ = \text{Im} \left( \frac{1-i}{2} (e^{2\pi} - 1) \right) = -\frac{e^{2\pi} + 1}{2} \quad \square. \]

Exercise 2. Using Euler’s formula, rederive the identities:

a) \( \sin(x + y) = \sin x \cos y + \cos x \sin y \).
b) \( \cos(x + y) = \cos x \cos y - \sin x \sin y \).

Solution:

We recall that for \( x, y \in \mathbb{R} \), one has:
\[ e^{i(x+y)} = e^{ix} \cdot e^{iy}. \]

We rewrite both sides by using Euler’s formula to obtain:
\[ \cos(x + y) + i \sin(x + y) = (\cos x + i \sin x) \cdot (\cos y + i \sin y). \]

It follows that:
\[ \cos(x + y) + i \sin(x + y) = (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \sin y \cos x). \]
Claims a) and b) now follow by taking real and imaginary parts of both sides \( \Box \).

**Exercise 3.** Find all complex numbers \( z \) such that:

\[ \begin{align*}
  a) & \ z^5 = 1. \\
  b) & \ z^7 = i. \\
  c) & \ Re(e^z) > 0.
\end{align*} \]

**Solution:**

a) The wanted complex numbers are \( z_k = e^{\frac{2\pi ik}{5}} = e^{\frac{ik\pi}{5}} = \cos\left(\frac{k\pi}{5}\right) + i\sin\left(\frac{k\pi}{5}\right), \) for \( k = 0, 1, \ldots, 5 \).

b) Since \( i = e^{\frac{i\pi}{2}} \), we can deduce that the solutions are given by \( z_k = e^{\frac{i\pi}{2} + \frac{2k\pi i}{5}}, \) for \( k = 0, 1, \ldots, 6 \).

c) We write \( z = re^{i\theta} \), where \( r > 0 \) and \( \theta \in [0, 2\pi) \). In this way, \( r \) and \( \theta \) are uniquely determined from \( z \). Since \( z = r\cos\theta + i\sin\theta \), we deduce that:

\[ e^z = r^e(\cos\theta + i\sin\theta) = e^r\cos\theta \cdot e^{ir\sin\theta} = e^r\cos\theta \cdot (\cos(r\sin\theta) + i\sin(r\sin\theta)) \]

Since \( e^r\cos\theta \) is a positive real number, the condition we need to satisfy is \( \cos(r\sin\theta) > 0 \). An equivalent way to write this is to say that there exists \( k \in \mathbb{Z} \) such that:

\[ r\sin\theta \in (\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi) \]. \( \Box \)

**Exercise 4.** a) For what \( c \in \mathbb{R} \) does there exist a non-zero function \( w : [0, 2\pi] \to \mathbb{C} \) such that:

\[ w'' - c^2w = 0 \]

and such that \( w(0) = w(2\pi) = 0 \)?

b) What if \( w \) instead solves \( w'' + c^2w = 0 \) (again with the assumption that \( w(0) = w(2\pi) = 0 \))?  

**Solution:**

Let us first suppose that \( c \neq 0 \). From ODE theory, we know that \( w = a_1e^{ct} + a_2e^{-ct} \) for some (complex numbers) \( a_1, a_2 \). The condition \( w(0) = w(2\pi) = 0 \) then implies that:

\[ \begin{align*}
  a_1 + a_2 &= 0 \\
  a_1e^{2\pi c} + a_2e^{-2\pi c} &= 0
\end{align*} \]

From the above two equations, it follows that \( a_1 = a_2 = 0 \) and so \( w \) is identically zero. If \( c = 0 \), then \( w = a_1 + a_2t \). In this case, \( w(0) = 0 \) implies that \( a_1 = 0 \) and \( w(2\pi) = 0 \) implies that \( a_2 = 0 \), and so \( w \) is again identically zero. Hence, in a), it is not possible to find such a function \( w \).

b) We now consider what happens when \( w'' + c^2w = 0 \). Based on part a), we need to assume that \( c \neq 0 \). In this case, we recall that \( w(t) = a_1\cos(ct) + a_2\sin(ct) \). Since \( w(0) = a_1 = 0 \), it follows that \( w(t) = a_2\sin(ct) \). We then obtain that \( w(2\pi) = a_2\sin(2\pi c) \). Since we want \( a_2 \neq 0 \) (since otherwise, \( w \) is identically zero), it follows that we need to have \( \sin(2\pi c) = 0 \), and hence \( 2\pi c = k\pi \) for some \( k \in \mathbb{Z} \). Consequently, \( c = \frac{k}{2} \) for some \( k \in \mathbb{Z} \setminus \{0\} \). \( \Box \)

**Exercise 5.** Suppose that \( w : [0, +\infty) \to \mathbb{R} \) solves the ODE:

\[ aw'' + bw' + cw = 0 \]

for some constants \( a, b, c \). Furthermore, we assume that \( b \geq 0 \).

a) Let us define the **Energy** to be:

\[ E(t) := \frac{1}{2}[a(w'(t))^2 + c(w(t))^2]. \]

Without solving the ODE (1), show that \( E'(t) \leq 0 \).

b) Under the additional assumption that \( a > 0 \) and \( c > 0 \), show that \( w(0) = 0 \) and \( w'(0) = 0 \) implies that \( w(t) = 0 \) for all \( t \geq 0 \).

c) Assume again that \( a > 0 \) and \( c > 0 \). Show that if \( w_1 \) and \( w_2 \) solve the ODE (1) and if \( w_1(0) = w_2(0) = 0 \)
$w_2(0), w'_1(0) = w'_2(0)$, then one can deduce that $w_1(t) = w_2(t)$ for all $t \geq 0$. In this way, we obtain uniqueness of solutions to (1).

Solution:

a) We use the product rule to calculate: $E'(t) = aw'w'' + cww'$. We can now use the ODE to deduce that $w'' = -bw' - cw$. Hence:

$$E'(t) = w'(-bw' - cw) + cww' = -b(w')^2 \leq 0$$

since $b \geq 0$. In other words, $E(t)$ is a decreasing function of $t$ on $[0, +\infty)$.

b) By assumption $E(0) = \frac{1}{2} \left[a(w'(0))^2 + c(w(0))^2\right] = 0$. Since $a, c > 0$, it follows that $E(t)$ is non-negative. Finally, from part a), it follows that $E(t)$ is a decreasing function on $[0, +\infty)$, hence $E(t)$ is identically zero on $[0, +\infty)$. In particular, since both $a$ and $c$ are positive, it follows that $w(t) = 0$ for all $t \geq 0$.

c) If $w_1$ and $w_2$ solve the ODE, then so does $w := w_1 - w_2$. The function $w$ then satisfies the conditions of part b) and the claim follows. □