Exercise 1. Suppose that \( u \) solves the boundary value problem:

\[
\begin{align*}
  u_t(x, t) - u_{xx}(x, t) &= 1, \quad 0 < x < 1, t > 0 \\
  u(x, 0) &= 0, \quad 0 \leq x \leq 1 \\
  u(0, t) = u(1, t) &= 0, \quad t > 0.
\end{align*}
\]

(a) Find a function \( v = v(x) \) which solves:

\[
\begin{align*}
  -v''(x) &= 1, \quad 0 < x < 1 \\
  v(0) = v(1) &= 0.
\end{align*}
\]

(b) Show that:

\[ u(x, t) \leq v(x) \]

for all \( x \in [0, 1], t > 0 \).

c) Show that:

\[ u(x, t) \geq (1 - e^{-2t})v(x) \]

for all \( x \in [0, 1], t > 0 \).

d) Deduce that, for all \( x \in [0, 1] \):

\[ u(x, t) \to v(x) \]

as \( t \to \infty \).

Solution:

(a) We need to solve \( v''(x) = -1 \) with boundary conditions \( v(0) = v(1) = 0 \). The ODE implies that \( v(x) = -\frac{1}{2}x^2 + Ax + B \) for some constants \( A, B \). We get the system of linear equations:

\[
\begin{align*}
  B &= 0 \\
  -\frac{1}{2} + A + B &= 0
\end{align*}
\]

from where it follows that:

\[ A = \frac{1}{2} \quad \text{and} \quad B = 0. \]

Hence:

\[ v(x) = \frac{1}{2} x \cdot (1 - x). \]

(b) Let us now think of \( v \) as a function of \( v \) as a function of \((x, t)\) which doesn’t depend on \( x \). By construction, we know that:

\[
\begin{align*}
  v_t(x, t) - v_{xx}(x, t) &= 1, \quad 0 < x < 1, t > 0 \\
  v(x, 0) &\geq 0, \quad 0 \leq x \leq 1 \\
  v(0, t) = v(1, t) &= 0, \quad t > 0.
\end{align*}
\]

Here, we used the fact that \( \frac{1}{2} x \cdot (1 - x) \geq 0 \) for \( 0 \leq x \leq 1 \). By using the Comparison principle for the heat equation (Exercise 3 on Homework Assignment 4), it follows that:

\[ u(x, t) \leq v(x, t) = v(x) \]
for all $x \in [0, 1], t > 0$.

c) Let us define:

$$w(x, t) := (1 - e^{-2t})v(x) = \frac{1}{2} (1 - e^{-2t}) \cdot x(1 - x)$$

We compute:

$$w_t(x, t) = e^{-2t} \cdot x(1 - x)$$

$$w_{xx}(x, t) = -(1 - e^{-2t}) = -1 + e^{-2t}.$$  

Hence:

$$w_t(x, t) - w_{xx}(x, t) = 1 - e^{-2t} \left(1 - x(1 - x)\right).$$

We know that for $x \in [0, 1]$, one has: $x(1 - x) \in [0, 1]$. Hence, it follows that:

$$w_t(x, t) - w_{xx}(x, t) \leq 1$$

for all $0 \leq x \leq 1$, $t > 0$. In particular, we deduce that:

$$\begin{cases}
    w_t(x, t) - w_{xx}(x, t) = 1, & \text{for } 0 < x < 1, t > 0 \\
    w(x, 0) = 0, & \text{for } 0 \leq x \leq 1 \\
    w(0, t) = w(1, t) = 0, & \text{for } t > 0.
\end{cases}$$

By using the comparison principle, it follows that, for all $x \in [0, 1], t > 0$, the following holds:

$$u(x, t) \geq w(x, t) = \frac{1}{2} (1 - e^{-2t}) \cdot x(1 - x) = (1 - e^{-2t})v(x).$$

d) Combining the results of parts b) and c), it follows that, for all $x \in [0, 1], t > 0$, it holds that:

$$(1 - e^{-2t})v(x) \leq u(x, t) \leq v(x).$$

Letting $t \to \infty$, it follows that:

$$u(x, t) \to v(x)$$

as $t \to \infty$. □

Exercise 2. a) Find the function $u$ solving (1) of the previous exercise by using separation of variables. Leave the Fourier coefficients in the form of an integral. [HINT: Consider the function $w := u - v$ for $u, v$ as in the previous exercise.]

b) Show that this is the unique solution of the problem (1).

c) By using the formula from part a), give an alternative proof of the fact that $u(x, t) \to v(x)$ as $t \to \infty$. In this part, one is allowed to assume that the Fourier coefficients at time zero are absolutely summable without proof.

Solution:

a) Let $\tilde{u}(x, t) := u(x, t) - \frac{1}{2}x(1 - x)$. Then the function $\tilde{u}$ solves:

$$\begin{cases}
    \tilde{u}_t(x, t) - \tilde{u}_{xx}(x, t) = 0, & \text{for } 0 < x < 1, t > 0 \\
    \tilde{u}(x, 0) = -\frac{1}{2}x(1 - x), & \text{for } 0 \leq x \leq 1 \\
    \tilde{u}(0, t) = \tilde{u}(1, t) = 0, & \text{for } t > 0.
\end{cases}$$

We look for $\tilde{u}$ in the form of a Fourier sine series with coefficients which depend on $t$.

$$\tilde{u}(x, t) = \sum_{n=1}^{\infty} A_n(t) \sin(n\pi x).$$
We first set \( t = 0 \) to deduce that:
\[
\tilde{u}(x,0) = -\frac{1}{2} x(1-x) = \sum_{n=1}^{\infty} A_n(0) \sin(n\pi x) = -\frac{1}{2} x(1-x).
\]
Hence, \( A_n(0) \) equals the \( n \)-th Fourier sine series coefficient of the function \(-\frac{1}{2} x(1-x)\) on \([0,1]\). In particular,
\[
A_n(0) = 2 \int_0^1 \left( -\frac{1}{2} x(1-x) \right) \sin(n\pi x) \, dx.
\]
In order for \( \tilde{u} \) to solve the heat equation, we need:
\[
A_n'(t) - n^2 \pi^2 A_n(t) = 0.
\]
Hence:
\[
A_n(t) = A_n(0) \cdot e^{-n^2 \pi^2 t}.
\]
Consequently:
\[
\tilde{u}(x,t) = \sum_{n=1}^{\infty} A_n(0) \cdot e^{-n^2 \pi^2 t} \cdot \sin(n\pi x).
\]
We then deduce that:
\[
u(x,t) = \frac{1}{2} x(1-x) + \sum_{n=1}^{\infty} A_n(0) \cdot e^{-n^2 \pi^2 t} \cdot \sin(n\pi x).
\]
b) Uniqueness of the problem (1) was shown in class by using the maximum principle and by using the energy method.
c) We note that:
\[
|u(x,t) - v(x)| = \left| \sum_{n=1}^{\infty} A_n(0) \cdot e^{-n^2 \pi^2 t} \cdot \sin(n\pi x) \right| \leq \sum_{n=1}^{\infty} |A_n(0)| \cdot e^{-n^2 \pi^2 t} \leq e^{-\pi^2 t} \cdot \sum_{n=1}^{\infty} |A_n(0)|.
\]
As is noted in the problem, we are allowed to assume that \( \sum_{n=1}^{\infty} |A_n(0)| < \infty \).

The claim now follows. \( \square \)

**Exercise 3.** Suppose that \( u : \mathbb{R}^3 \to \mathbb{R} \) is a harmonic function.

a) By using the Mean Value Property (in terms of averages over spheres), show that, for all \( x \in \mathbb{R}^3 \), and for all \( R > 0 \), one has:
\[
u(x) = \frac{3}{4\pi R^3} \int_{B(x,R)} u(y) \, dy.
\]
b) Suppose, moreover, that \( \int_{\mathbb{R}^3} |u(y)| \, dy < \infty \). Show that then, one necessarily obtains:
\[
u(x) = 0
\]
for all \( x \in \mathbb{R}^3 \).

---

\( ^1 \) We can integrate by parts twice in the definition of \( A_n(0) \) and use the fact that \(-\frac{1}{2} x(1-x)\) vanishes at \( x = 0 \) and \( x = 1 \) in order to deduce that:
\[
|A_n(0)| \leq \frac{C}{n^2},
\]
from where it indeed follows that \( \sum_{n=1}^{\infty} |A_n(0)| < \infty \).
Solution:

a) Let us fix \( x \in \mathbb{R}^3 \). The Mean Value Property, proved in Exercise 1 of Homework Assignment 7, implies that, for all \( r > 0 \):

\[
(2) \quad u(x) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y) dS(y).
\]

We note that:

\[
\frac{3}{4\pi R^3} \int_{B(x,R)} u(y) dS(y) = \frac{3}{4\pi R^3} \int_0^R \left( \int_{\partial B(x,r)} u(y) dS(y) \right) dr.
\]

By the Mean Value Property (2), it follows that this expression equals:

\[
\frac{3}{4\pi R^3} \int_0^R 4\pi r^2 u(x) \, dr = u(x) \cdot \frac{3}{4\pi R^3} \cdot \int_0^R 4\pi r^2 \, dr = u(x).
\]

b) We note that, by part a), it follows that:

\[
|u(x)| \leq \frac{3}{4\pi R^3} \int_{B(x,R)} |u(y)| dy \leq \frac{3}{4\pi R^3} \int_{\mathbb{R}^3} |u(y)| dy.
\]

Since \( \int_{\mathbb{R}^3} |u(y)| \, dy < \infty \), we can let \( R \to \infty \) to deduce that \( |u(x)| = 0 \). It follows that \( u \) is identically equal to zero. \( \square \)

Exercise 4. Suppose that \( u : B(0,2) \to \mathbb{R} \) is a harmonic function on the open ball \( B(0,2) \subseteq \mathbb{R}^2 \), which is continuous on its closure \( \overline{B(0,2)} \). Suppose that, in polar coordinates:

\[
u(2, \theta) = 3 \sin 5\theta + 1\]

for all \( \theta \in [0, 2\pi] \).

a) Find the maximum and minimum value of \( u \) in \( \overline{B(0,2)} \) without explicitly solving the Laplace equation.

b) Calculate \( u(0) \) without explicitly solving the Laplace equation.

Solution:

a) By using the weak maximum principle for solutions to the Laplace equation, we know that the maximum of the function \( u \) on \( \overline{B(0,2)} \) is achieved on \( \partial B(0,2) \). We observe that the function \( u(2, \theta) = 3 \sin 5\theta + 1 \) takes values in \([-2, 4]\). It equals \(-2\) when \( \sin 5\theta = -1 \), which happens at \( \theta = \frac{2\pi}{10} \) (for example). Moreover \( u(2, \theta) = 4 \) when \( \sin 5\theta = 1 \), which happens at \( \theta = \frac{\pi}{10} \) (for example). Hence, the maximum value of \( u \) on \( \overline{B(0,2)} \) is 4 and the minimum value of \( u \) on \( B(0,2) \) is \(-2\).

b) We use the Mean Value Property to deduce that \( u(0) \) equals the average of \( u \) over the circle \( \partial B(0,2) \). Since the average of the \( 3 \sin 5\theta \) term equals zero, it follows that \( u(0) = 1 \). \( \square \)