Exercise 1. (A mean value formula for the Poisson equation on $B(0,r)$) [Evans, Problem 3 in Chapter 2] Suppose that $r > 0$, and that $n \geq 3$. Consider $B(0,r) \subseteq \mathbb{R}^n$ and $u \in C^2(B(0,r)) \cap C(B(0,r))$ such that:

\begin{align*}
\begin{cases}
-\Delta u = f, \text{ on } B(0,r) \\
u = g, \text{ on } \partial B(0,r) 
\end{cases}
\end{align*}

(1) Modify the proof of the mean value formula for harmonic functions to show that:

$$u(0) = \int_{\partial B(0,r)} g(y) dS(y) + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left( \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f(x) dx$$

Exercise 2. (A direct proof of the maximum principle) [Evans, Problem 4 in Chapter 2] Suppose that $U \subseteq \mathbb{R}^n$ is open and bounded and suppose that $u \in C^2(U) \cap C(\overline{U})$ is harmonic on $U$. By considering the functions $u_\epsilon := u + \epsilon |x|^2$, for $\epsilon > 0$, show that:

$$\max_U u = \max_{\partial U} u.$$

Exercise 3. (Subharmonic functions) [Evans, Problem 5 in Chapter 2] We say that $v \in C^2(U)$ is subharmonic if:

$$-\Delta v \leq 0.$$

(1) Prove that for $v$ subharmonic, one has:

$$v(x) \leq \frac{1}{B(x,r)} \int_{B(x,r)} v(y) dy$$

for all $B(x,r) \subseteq U$.

(2) Show that $\max_{B} v = \max_{\partial U} v$.

(3) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume that $u$ is harmonic and take $v := \phi(u)$. Prove that $u$ is subharmonic.

(4) Prove that $v := |\nabla u|^2$ is subharmonic whenever $u$ is harmonic.

Exercise 4. (A specific class of harmonic functions) Find all polynomials $P(x,y) = \sum_{k=0}^{n} c_k x^k y^{n-k}$ in $x,y$ which are homogeneous of degree $n$ and which are harmonic. Here, $c_k \in \mathbb{C}$.

Each problem is worth 5 points. This assignment is due in class on Wednesday, September 28. Good Luck!