Exercise 1. (An interpolation inequality) [5 points]

Suppose that $1 \leq p, q \leq \infty$ and suppose that $\theta \in [0,1]$ is given. Define $r$ by:

$$\frac{1}{r} := \frac{\theta}{p} + \frac{1-\theta}{q}$$

Show that:

$$\|f\|_{L^r} \leq \|f\|_{L^p}^\theta \|f\|_{L^q}^{1-\theta}.$$

Exercise 2. (More on the Hausdorff-Young Inequality) [15 points]

a) Suppose that $1 \leq p, q \leq \infty$ are such that there exists $C > 0$ with the property that:

$$\|\hat{f}\|_{L^q} \leq C \|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$. Show that $\frac{1}{p} + \frac{1}{q} = 1$. (HINT: Use scaling).

b) Suppose that $1 \leq p \leq \infty$ is such that there exists $C > 0$ with the property that:

$$\|\hat{f}\|_{L^{p'}} \leq C \|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$. Here $p'$ denotes the Hölder conjugate of $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Show that necessarily $1 \leq p \leq 2$.

HINT: This is a subtle construction. The idea is that, given $N \in \mathbb{N}$, one defines the function:

$$f_N(x) := \sum_{n=1}^N e^{2\pi i x \cdot (nv)} g(x - nv)$$

for $g$ a Gaussian and $v \in \mathbb{R}^n$, a unit vector.

i) How is the Fourier transform of $f_N$ related to $f_N$?

ii) What is a good lower bound for $\|f_N\|_{L^2}$? (It is good to look at parts of $f_N$ near $xv$.)

iii) What is an upper bound for $\|f_N\|_{L^1}$ and for $\|f_N\|_{L^\infty}$?

iv) Use iii) and Exercise 1 to deduce an upper bound for $\|f_N\|_{L^p}$.

v) Is it possible to deduce that $\|f_N\|_{L^p} \sim N^r$ for some power $r$? How about $\|\hat{f_N}\|_{L^{p'}}$?

Exercise 3. (Generalized Young’s Inequality) [5 points]

Suppose that $1 \leq p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. By using the Riesz-Thorin Interpolation Theorem, show that:

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}.$$

Exercise 4. (A refinement of Young’s inequality when $r = \infty$) [5 points]

Suppose that $1 < p < \infty$ and suppose that $f \in L^p(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$. Show that $f * g$ is uniformly continuous and that it decays to zero at infinity.

HINT: Recall that $\lim_{t \to 0} \|f(\cdot + t) - f\|_{L^p} = 0$.

This homework assignment is due in class on Wednesday, November 30. Good Luck!