Adelic and local $L^1$-isomorphisms

Let $K$ be a number field or a non-archimedean local field of characteristic zero. Let $G$ be a linear algebraic group over $Q$.

Definition. The finite-adelic resp. local Hecke algebra is

\[
\mathcal{H}_G(K) = \begin{cases} 
\mathcal{C}^\infty(G(A_K), R) & \text{if } K \text{ is global} \\
\mathcal{C}^\infty(G(K), \mathbb{C}) & \text{if } K \text{ is local}.
\end{cases}
\]

The (left) invariant Haar measure induces an $L^1$-norm on the Hecke algebras. An $L^1$-isomorphism between Hecke algebras is an algebra isomorphism which is an isometry for the $L^1$-norm.

Theorem A. Let $K$ and $L$ be two number fields, respectively two non-archimedean local fields of characteristic zero. There is an $L^1$-isomorphism of Hecke algebras $\mathcal{H}_G(K) \cong_{L^1} \mathcal{H}_G(L)$ if and only if there is

\[
\begin{cases} 
\text{a ring isomorphism } A_K \cong A_L & \text{if } K \text{ is global} \\
\text{a field isomorphism } K \cong L & \text{if } K \text{ is local}.
\end{cases}
\]

N.B. These isomorphisms are topological.

Proof. Denote

\[ G(K) = \begin{cases} 
G(A_K) & \text{if } K \text{ is global} \\
G(K) & \text{if } K \text{ is local}.
\end{cases} \]

Stone-Weierstrass yields that $\mathcal{H}_G(K)$ is dense in $C_0(G(K))$, hence in $L^1(G(K))$, so $\mathcal{H}_G(K) \cong_{L^1} \mathcal{H}_G(L)$ induces an isometry $L^1(G(K)) \cong_{L^1} L^1(G(L))$. Now $L^1(G(K)) \cong_{L^1} L^1(G(L))$ implies that $G(K) \cong G(L)$ by results of Wenzel [4].

For $K$ and $L$ local, it is a classical result that $G(K) \cong G(L)$ implies that $K \cong L$. For $K$ and $L$ global, this follows from Theorem B.

Local Morita equivalences

Let $K$ be a non-archimedean local field of characteristic zero and let $G = GL_2$.

Theorem C. There is a Morita equivalence

\[ \mathcal{H}_G(K) \sim_M \left( \bigoplus_N \mathbb{C}[T,Y^{-1}] \right) \oplus \left( \bigoplus_N \mathbb{C}[X,Y^{-1}] \right) \oplus \left( \bigoplus_N \mathbb{C}[Z^2 \times S_2] \right) \]

Proof. We use the Bernstein decomposition

\[ \mathcal{H}_G(K) = \bigoplus_{\rho \in R(G)(K)} \mathcal{H}_G^\rho(K) \]

where

\[ \mathcal{H}_G^\rho(K) = \mathcal{H}_G(K) \ast e_\rho \ast \mathcal{H}_G(K) \sim_M e_\rho \ast \mathcal{H}_G(K) \ast e_\rho \]

for some idempotent $e_\rho$ depending on a representation $\rho: H \to End_{\mathbb{C}}(W)$ of a compact open subgroup $H$ of $G(K)$, and

\[ e_\rho \ast \mathcal{H}_G(K) \ast e_\rho \cong \mathcal{H}(G(K), \rho) \otimes_{\mathbb{C}} End_{\mathbb{C}}(W), \]

where $\mathcal{H}(G(K), \rho)$ is the $\rho$-spherical or intertwining algebra for $\rho$. Now,

\[ \mathcal{H}(G(K), \rho) \cong \begin{cases} 
\mathbb{C}[T,T^{-1}] & \text{if } \rho \text{ is supercuspidal} \\
\mathbb{C}[X,Y^{-1}] & \text{if } K \text{ is a non-special principal series} \\
\mathbb{C}[Z^2 \times S_2] & \text{if } \rho \text{ is a special rep. or finite-dim.}
\end{cases} \]

of which occurs countably infinitely many times.

Corollary D. Let $K$ and $L$ be any non-arch. local fields of char. zero. Then $\mathcal{H}_G(K) \sim_M \mathcal{H}_G(L)$.

Adelic point group isomorphisms.

Let $K$ be a number field and $G$ a linear algebraic group over $Q$.

Definition. $G$ is fertile for $K$ if it is $K$-split and its connected component of the identity is not a direct product $T \times U$ of a torus and a unipotent group.

Theorem B. Let $K$ and $L$ be number fields and let $G/Q$ be fertile for $K$ and $L$. There is a topological group isomorphism $G(A_K) \cong G(A_L)$ if and only if there is a topological ring isomorphism $A_K \cong A_L$.

Proof. Step 1: Let $U$ be a fixed maximal unipotent subgroup of $G$. Then any maximal divisible subgroup $D$ of $G(A_K)$ is conjugate to $U(A_K)$, characterising $U(A_K)$ group theoretically inside $G(A_K)$.

Step 2:

- Let $N = N_G(A_K)$ be the normaliser of $D$.
- Let $\mathbb{V} = [N, \mathbb{D}] / [\mathbb{D}, \mathbb{D}] \leq \mathbb{D}^{ab}$.
- Let $T = N/\mathbb{D}$. It acts on $\mathbb{V}$ through say $\ell$ distinct non-trivial characters. Then $Z(End_{\mathbb{V}}(V)) \cong A_K^{\ell}$.

Step 3: Forming quotients $A_K / \mathfrak{m}$ by maximal ideals yields the multiset $\{K_p, p \in M_K^{\text{finite}}\}$, where $M_K^{\text{finite}}$ is the set of finite places of $K$.

Step 4: Doing this construction for both $K$ and $L$ proves that $K$ and $L$ are locally isomorphic, i.e. $A_K \cong A_L$. Then $A_K \cong A_L$ by results of Klingen [3].

Remark. The proof goes through for abstract isomorphisms instead of topological isomorphisms.

Remarks and references

Remark 1. Let $K$ and $L$ be number fields. The additive groups $(A_K, +)$ and $(A_L, +)$ are isomorphic if and only if $[K : Q] = [L : Q]$. However, if this isomorphism is local, then $K$ and $L$ are arithmetically equivalent.

Remark 2. Let $K$ and $L$ be imaginary quadratic number fields, different from $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$. Then the multiplicative groups $A_K^\times$ and $A_L^\times$ are isomorphic. Again, if this isomorphism is local, then $K$ and $L$ are arithmetically equivalent.

Remark 3. Remarks 1 and 2 show why we need fertility in Theorem B.

References

2. V. Karemker, Structure of Hecke algebras over local fields, in preparation.

2015 Mathematisch Instituut, Universiteit Utrecht, The Netherlands
V.Z.Karemker@uu.nl