Finiteness Properties of Difference Algebraic Groups

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Difference algebra

Definition

A *difference ring (σ-ring)* is ring R together with a ring endomorphism $\sigma\colon R \to R$.

k a σ-field, e.g., $k = \mathbb{C}(x)$ with $\sigma(f(x)) = f(x+1)$. The σ-polynomial ring over k is

$$k\{y\} = k\{y_1, \ldots, y_n\} = k[y_1, \ldots, y_n, \sigma(y_1), \ldots, \sigma(y_n), \sigma^2(y_1), \ldots].$$

$F \subset k\{y\}$, R a k-σ-algebra

$$\mathbb{V}_R(F) = \{a \in R^n |\ f(a) = 0\ \forall\ f \in F\}$$

Definition

A functor of the form $R \rightsquigarrow \mathbb{V}_R(F)$ is called a *σ-variety*.

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Difference algebraic groups

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A *σ-algebraic group G* is a group object in the category of σ-varieties.

Examples:

$$G(R) = \{g \in R^\times | \; g\sigma^2(g)^3 = 1\} \leq \mathbb{G}_m(R)$$

$$G(R) = \{g \in R | \; \sigma^n(g) + \lambda_{n-1}\sigma^{n-1}(g) + \ldots + \lambda_0 y = 0\} \leq \mathbb{G}_a(R)$$

$$G(R) = \{g \in \mathrm{GL}_n(R) | \; g\sigma(g) = \sigma(g)g = I_n\} \leq \mathrm{GL}_n(R)$$

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Difference algebraic groups

Facts:

- The category of σ-varieties is anti–equivalent to the category of finitely σ-generated k-σ-algebras.
- The category of σ-algebraic groups is ant–equivalent to the category of finitely σ-generated k-σ-Hopf algebras.

$$G \leftrightarrow k\{G\}$$

A σ-closed subgroup H of G corresponds to a σ-Hopf ideal $\mathbb{I}(H) \subset k\{G\}$, i.e., a Hopf ideal with $\sigma(\mathbb{I}(H)) \subset \mathbb{I}(H)$. If there exists $B \subset \mathbb{I}(H)$ finite with

$$\mathbb{I}(H) = [B] = (B, \sigma(B), \ldots)$$

we say that $\mathbb{I}(H)$ is *finitely σ-generated*.

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Three finiteness theorems

Let $k\{G\}$ be a finitely σ-generated k-σ-Hopf algebra.

Theorem 1

Every σ-Hopf ideal of $k\{G\}$ is finitely σ-generated.

Theorem 2

Every k-σ-Hopf subalgebra of $k\{G\}$ is finitely σ-generated.

Theorem 3

There are only finitely many minimal prime σ-ideals in $k\{G\}$.

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Geometric corollaries

Corollary to Theorem 1

Every descending chain of σ-closed subgroups of a σ-algebraic group is finite.

Corollay to Theorem 2

If $N \trianglelefteq G$ is a normal σ-closed subgroup, the quotient G/N is σ-algebraic.

Corollary to Theorem 3

A σ-algebraic group has only finitely many σ-components.

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Open questions

A σ-ideal $\mathfrak{a}$ is called *mixed* if $ab \in \mathfrak{a}$ implies $a\sigma(b) \in \mathfrak{a}$.

Conjecture 1 (Hrushovski)

Every ascending chain of radical mixed σ-ideals in a finitely σ-generated k-σ-algebra is finite.

Conjecture 2

A finitely σ-generated k-σ-algebra has only finitely many minimal prime σ-ideals.

Conjecture 1 and Conjecture 2 are equivalent. Theorem 3 proves Conjecture 2 for k-σ-Hopf algebras.

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Applications of Theorem 1

Chevalley's theorem for σ-algebraic groups

Every σ-closed subgroup of a σ-algebraic group G is the stabilizer of a line in a finite dimensional representation of G.

Dimension theorem for σ-algebraic groups

Let $H_1, H_2 \leq G$ be σ-closed subgroups. Then

$$\sigma\text{-}\dim(H_1 \cap H_2) \geq \sigma\text{-}\dim(H_1) + \sigma\text{-}\dim(H_2) - \sigma\text{-}\dim(G).$$

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Proofs

Theorem 1

Every σ-Hopf ideal of $k\{G\}$ is finitely σ-generated.

Theorem 2

Every k-σ-Hopf subalgebra of $k\{G\}$ is finitely σ-generated.

Theorem 3

There are only finitely many minimal prime σ-ideals in $k\{G\}$.

Theorem 1 $\Rightarrow$ Theorem 2.
Theorem 3:

$$\sigma\text{-components of } G \leftrightarrow \sigma\text{-components of } G/G^o$$

Proof of Theorem 1, Zariski closures

W.l.o.g G is a σ-closed subgroup of GL_n. We have to show that $\mathbb{I}(G) \subset k\{\mathrm{GL}_n\} = k\{X, \frac{1}{\det(X)}\}$ is finitely σ-generated. For $i \geq 0$ the ideal

$$\mathbb{I}(G) \cap k[X, 1/\det(X), \ldots, \sigma^i(X), 1/\det(\sigma^i(X))]$$

defines an algebraic subgroup $G[i]$ of GL_n^{i+1}. We have morphisms

$$\pi_i \colon G[i] \to G[i-1],\ (g_0, \ldots, g_i) \mapsto (g_0, \ldots, g_{i-1})$$

and

$$\sigma_i \colon G[i] \to {}^{\sigma}(G[i-1]),\ (g_0, \ldots, g_i) \mapsto (g_1, \ldots, g_i).$$

Set

$$\mathcal{G}_i = \ker(\pi_i).$$

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Proof of Theorem 1, the growth group

Proposition

There exists $m \geq 0$ sucht that $\sigma_i \colon \mathcal{G}_i \longrightarrow {}^{\sigma}\mathcal{G}_{i-1}$ is an isomorphism for $i > m$.

The algebraic group $\mathcal{G} := \mathcal{G}_m$ (with m minimal) is called the *growth group* of G (w.r.t to $G \hookrightarrow \mathrm{GL}_n$).

$$\sigma\text{-}\dim(G) := \dim(\mathcal{G})$$

and

$$\mathrm{ld}(G) := |\mathcal{G}| := \dim_k k[\mathcal{G}]$$

do not depent on $G \hookrightarrow \mathrm{GL}_n$. One can show that $\mathbb{I}(G)$ is σ-generated by $\mathbb{I}(G) \cap k[X, 1/\det(X), \ldots, \sigma^m(X), 1/\det(\sigma^m(X)]$.

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Thank you!