Galois theory of strongly normal differential and difference extensions

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Outline

The strongly normal property

Assume: Char 0, $\Delta = \{\delta\}$, constants algebraically closed

A finite field extension $L|K$ is normal (=Galois) if for every field extension $M|K$ and every pair of K-morphisms $\tau_s, \tau_t \colon L \to M$ we have $\tau_t(L) \subset \tau_s(L)$.

Definition

Let $L|K$ be a finitely generated extension of δ-fields with $L^\delta = K^\delta$. Then $L|K$ is strongly normal if

$$\tau_t(L) \subset \tau_s(L) M^\delta$$

for every pair $\tau_s, \tau_t \colon L \to M$ of K-δ-morphisms with M a δ-field extension of K.

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Definition

Let $L|K$ be a finitely generated extension of δ-fields with $L^\delta = K^\delta$. Then $L|K$ is strongly normal/δ-normal if

$$\tau_t(L) \subset \tau_s(L) M^\delta$$

for every pair $\tau_s, \tau_t \colon L \to M$ of K-δ-morphisms with M a δ-field extension of K/K-δ-algebra.

The Galois group as an algebraic group

How to get $\mathrm{Aut}^{\delta}(L|K)$ as an algebraic group?

1. Weil's group chunk theorem (Kolchin, Bialynicki-Birula, Umemura, Moreno,...)
2. Axiomatic algebraic groups (Kolchin, Okugawa, Infante)
3. Differential schemes (Kovacic)

Key ideas of Kovacic's approach

For a δ-scheme $(X, \mathcal{O}_X)$ define its constants $(X^\delta, \mathcal{O}_{X^\delta})$ by $X^\delta := X$ and $\mathcal{O}_{X^\delta}(U) = \mathcal{O}_X(U)^\delta$.

For $L|K$ strongly normal

$$\mathcal{G} := (\delta\text{-}\operatorname{Spec}(L \otimes_K L))^\delta$$

is a scheme (of finite type over constants $k = K^\delta = L^\delta$). It inherits a group structure from the groupoid structure of $X := \delta\text{-}\operatorname{Spec}(L \otimes_K L)$. X is split, i.e., $X \simeq \delta\text{-}\operatorname{Spec}(L) \times_k \mathcal{G}$.

δ-geometric characterization of strongly normal extensions

Let $L|K$ be finitely δ-generated with $L^\delta = K^\delta$. Then $L|K$ is strongly normal iff $\delta\text{-}\operatorname{Spec}(L \otimes_K L)$ is split.

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An intrinsic characterization of PV-extensions

Theorem (W.)

Let $L|K$ be finitely δ-generated with $L^\delta = K^\delta$. Then $L|K$ is PV if and only if $L|K$ is δ-normal.

Difference equations

A σ-ring is a ring R with an endomorphism $\sigma\colon R \to R$.

Assume: Characteristic arbitrary, constants algebraically closed

PV-theory: Cannot always find a Picard-Vessiot σ-*field* extension $L|K$ for $\sigma(y) = Ay$. But $L = L_1 \times \cdots \times L_d$, with L_i fields and $\sigma(L_i) \subset L_{i+1}$ is possible. We call such L a σ-*pseudo field* (=total, Noetherian, σ-simple σ-ring).

$$\sigma\text{- Spec}(R) := \{\mathfrak{q} \in \text{Spec}(R) |\ \sigma^{-1}(\mathfrak{q}) = \mathfrak{q}\}$$

$$\sigma^{\infty}\text{- Spec}(R) := \{\mathfrak{q} \in \text{Spec}(R) |\ \exists\ d \geq 1 : \sigma^{-d}(\mathfrak{q}) = \mathfrak{q}\}$$

If $\mathfrak{q}$ is σ^d-prime then $L := k(\sigma^{d-1}(\mathfrak{q})) \times \cdots \times k(\mathfrak{q})$ is a σ-pseudo field.

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Strongly normal extensions of σ-pseudo fields

Definition

Let $L|K$ be a finitely generated extension of σ-pseudo fields with $L^\sigma = K^\sigma$. Then $L|K$ is *strongly normal* if

- For every pair $\tau_s, \tau_t \colon L \to M$ of K-σ-morphisms into some σ-pseudo field extension of M of K we have $\tau_t(L) \subset \tau_s(L)M^\sigma$.
- There exists a σ-overring M of $L \otimes_K L$ such that $\tau_t(L) \subset \tau_s(L)M^\sigma$, where $\tau_s(a) = a \otimes 1$ and $\tau_t(a) = 1 \otimes a$.

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How to get $\mathrm{Aut}^\sigma(L|K)$ as algebraic group?

Define $\mathrm{Gal}(L|K)$ as a functor from schemes of finite type over $k = K^\sigma = L^\sigma$ to groups by

$$\mathrm{Gal}(L|K)(Y) = \mathrm{Aut}(L \times_k Y | K \times_k Y).$$

Then $\mathrm{Gal}(L|K)$ is represented by $\mathcal{G} := (\sigma^\infty\text{- Spec}(L \otimes_K L))^\sigma$. For X a "σ^∞-scheme", X^σ :=orbits of σ on X,

$$\pi : X \to X^\sigma$$

$\mathcal{O}_{X^\sigma}(U) := \mathcal{O}_X(\pi^{-1}(U))^\sigma$ for $U \subset X^\sigma$ open.

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Characterization of strongly normal σ-extensions

Let $L|K$ be a finitely generated σ-separable extension of σ-pseudo fields with $L^\sigma = K^\sigma (= k)$. Then the following are equivalent:

1. The functor $\mathrm{Gal}(L|K)$ is representable by a group scheme $\mathcal{G}$, and $Z = \sigma^\infty\text{-}\operatorname{Spec}(L)$ is a $\mathcal{G}$-torsor.
2. $\sigma^\infty\text{-}\operatorname{Spec}(L \otimes_K L)$ is split, i.e., there exists a scheme $\mathcal{G}$ of finite type over $k = K^\sigma$ such that $\sigma^\infty\text{-}\operatorname{Spec}(L \otimes_K L) \simeq L \times_k \mathcal{G}$.
3. $L|K$ is strongly normal.

Thank you!