A note on the “3x + 1” problem- H. Wilf, March 19, 1993

Take a positive integer \( n \) and repeat the following operations: if \( n \) is odd then replace \( n \) by \( 3n + 1 \), whereas if \( n \) is even then replace \( n \) by \( n/2 \). For example if we start with \( n = 34 \) we obtain successively,

\[
34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1.
\]

The problem is to prove or disprove the assertion that *whatever n you start with, you’ll end up at n = 1 after a finite number of steps*. This problem has been unsolved since it was proposed in the 1930’s.

If \( n \geq 2 \) is an integer, then \( S(n) \), the successor of \( n \), is \( n/2 \) if \( n \) is even and is \( 3n + 1 \) if \( n \) is odd. Say that \( m \) is a good integer if the chain

\[
m \to S(m) \to S(S(m)) \to \cdots,
\]

that begins with \( m \) and proceeds by the successor operator, reaches 1 after finitely many steps. Such a chain, that begins with \( m \) and reaches 1 in finitely many steps, will be called an \( m \)-chain.

Define \( \epsilon_m \) to be 1 if \( m = 1 \) or if \( m \geq 2 \) and \( m \) is good, and to be 0 otherwise. The 3x + 1 problem is to prove or disprove the assertion that \( \epsilon_m = 1 \) for every positive integer \( m \).

We have the following three relations among the \( \epsilon \)'s:

(a) \( \epsilon_1 = 1 \), and

(b) \( \epsilon_{2k} = \epsilon_k \) for all \( k \geq 1 \), and

(c) \( \epsilon_{2k+1} = \epsilon_{6k+4} \) for all \( k \geq 1 \).

Indeed, if \( \epsilon_{2k} = 1 \), there is a \( 2k \)-chain \( 2k \to k \to \cdots \), so \( \epsilon_k = 1 \) also. Conversely if \( \epsilon_k = 1 \) there is a \( k \) chain, and therefore a \( 2k \)-chain also since \( S(2k) = k \), which proves (b). Similarly we prove (c).

Next define the generating function \( f(t) = \sum_{k \geq 1} \epsilon_k t^k \). To discover the functional equation that \( f \) satisfies, we do the following: multiply (a) above by \( t \), add to the sum over \( k \geq 1 \) of the result of multiplying (b) by \( t^{2k} \), and add the sum over \( k \geq 1 \) of the result of multiplying (c) by \( t^{2k+1} \). The result is that

\[
f(t) = t + \sum_{k \geq 1} \epsilon_k t^{2k} + \sum_{k \geq 1} \epsilon_{6k+4} t^{2k+1} = f(t^2) + \sum_{k \geq 0} \epsilon_{6k+4} t^{2k+1}.
\]

This is the first form of the functional equation. From this form it is easy to check that \( f(t) = t/(1 - t) \) satisfies the equation. It follows that the 3x + 1 problem has an affirmative
answer if and only if equation (1) has a unique solution in the class of power series of the form $f(t) = t + O(t^2)$.

For a second form of the functional equation (1) we observe that for any power series $g(t) = \sum_{n\geq0} b_n t^n$, we can pick out the subseries in which the exponents are congruent to $u$ modulo $v$ by means of the easily verified identity

$$\sum_{r \geq 0} b_{rv+u} t^{rv+u} = \frac{1}{v} \sum_{\omega^v=1} \omega^{-u} g(\omega t),$$

in which the sum extends over the $v$-th roots of unity.

Now if we replace $t$ by $t^3$ in (1) we obtain

$$f(t^3) = f(t^6) + \frac{1}{t} \sum_{k \geq 0} \epsilon_{6k+4} t^{6k+4} = f(t^6) + \frac{1}{6t} \sum_{\omega^6=1} \omega^2 f(\omega t),$$

in which the sum is over the sixth roots of unity. Thus-

The 3x + 1 problem has an affirmative answer iff the functional equation

$$f(t^3) = f(t^6) + \frac{1}{6t} \sum_{\omega^6=1} \omega^2 f(\omega t)$$

has a unique solution in the class of functions $f(t)$ s.t. $f(0) = 0$, $f'(0) = 1$, and $f(t) = \sum_{j \geq 1} \epsilon_j t^j$, ($\forall j: \epsilon_j = 0, 1$).

A proposition that is somewhat stronger than the 3x+1 problem is therefore the following.

Let $f$ satisfy (3) and be holomorphic in the unit disk, with $f(0) = f'(0) = 0$. Then $f$ vanishes identically.

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Suppose we represent $f(z)$ by the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

If we substitute this into (3) we get

$$\frac{1}{2\pi i} \int_C f(\zeta) d\zeta \left\{ \frac{1}{\zeta - z^3} - \frac{1}{\zeta - z^6} - \frac{1}{6z} \sum_{\omega^6=1} \frac{\omega^2}{\zeta - \omega z} \right\} = 0.$$

But it’s easy to check that the inner sum is

$$\sum_{\omega^6=1} \frac{\omega^2}{\zeta - \omega z} = \frac{6z^4 \zeta}{\zeta^6 - z^6}.$$
To sum up, let’s define the kernel function

\[ K(\zeta, z) = \frac{z^3 - z^6}{(z^3 - \zeta)(z^6 - \zeta)} - \frac{z^3 \zeta}{\zeta^6 - z^6}, \]

and the integral operator

\[ (Kf)(z) = \frac{1}{2\pi i} \int_C K(\zeta, z)f(\zeta)d\zeta, \]

from \( H \to H \), where \( H \) is the set of holomorphic functions on the unit disk which vanish at the origin.

The \( 3x+1 \) problem is equivalent to the assertion that the operator \( K \) has a one-dimensional kernel, namely the constant multiples of \( z/(1 - z) \).