SOME APPLICATIONS OF THE INEQUALITY OF ARITHMETIC AND GEOMETRIC MEANS TO POLYNOMIAL EQUATIONS

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The purpose of this note is to point out a simple generalization of the inequality

$$(z_1 z_2 \cdots z_n)^{1/n} \leq \frac{1}{n} (z_1 + \cdots + z_n)$$

of arithmetic and geometric means, which will hold when the arguments of the complex numbers $z_1, \cdots, z_n$ are suitably restricted. We shall apply the resulting inequality to the roots of polynomial equations, obtaining first a quantitative form of the Gauss-Lucas theorem, and then some relationships between the coefficients of a polynomial and the size of a sector containing its roots.

1. The inequality. The basic result is

**Theorem 1.** Suppose

$$|\arg z_i| \leq \psi < \frac{\pi}{2}, \quad i = 1, 2, \cdots, n.$$  

Then

$$(1) \quad |z_1 z_2 \cdots z_n|^{1/n} < \left( \sec \psi \right) \frac{1}{n} \left| z_1 + z_2 + \cdots + z_n \right|$$

unless $n$ is even and $z_1 = \cdots = z_{n/2} = \overline{z}_{(n/2)+1} = \cdots = \overline{z}_n = r e^{i\psi}$, in which case equality holds.

**Proof.** We have

$$|z_1 + z_2 + \cdots + z_n| \geq \left| \text{Re}(z_1 + \cdots + z_n) \right|$$

$$= \left( |z_1| \cos \phi_1 + |z_2| \cos \phi_2 + \cdots \right.$$  

$$\left. + |z_n| \cos \phi_n \right)$$

$$(2) \quad \geq \left( \cos \psi \left( |z_1| + \cdots + |z_n| \right) \right)$$

$$\geq n \cos \psi \left( |z_1| + |z_2| + \cdots + |z_n| \right)^{1/n}$$

Received by the editors March 2, 1962.
as claimed. All signs of equality hold only when
(a) \( \text{Im} (z_1 + \cdots + z_n) = 0 \)
(b) \( \cos \phi_i = \cos \psi \) \((i = 1, 2, \cdots, n)\)
(c) \( |z_1| = |z_2| = \cdots = |z_n| \)

which imply the configuration stated in the theorem. For odd \( n \) the constant \( \sec \psi \) is only asymptotically best possible.

2. Application to polynomials. Let

(4) \( P(z) = a_0 + a_1z + \cdots + a_nz^n = a_n(z - z_1) \cdots (z - z_n) \)

be given and let \( K \) denote the convex hull of the zeros \( z_1, \cdots, z_n \) of \( P(z) \). Let \( z \) be outside \( K \), and suppose that, from \( z, K \) subtends an angle \( 2\psi \). Then the spread in the arguments of the numbers

\[
\frac{1}{z - z_1}, \ldots, \frac{1}{z - z_n}
\]

is at most \( 2\psi \), and from Theorem 1,

\[
\left| \frac{1}{(z - z_1)} \cdots \frac{1}{(z - z_n)} \right|^{1/n} \leq \left( \sec \psi \right)^{-1/n} \frac{1}{n} \left| \sum_{j=1}^{n} \frac{1}{z - z_j} \right|.
\]

But this is just the assertion that

\[
\left| \frac{a_n}{P(z)} \right|^{1/n} \leq \frac{\sec \psi}{n} \left| \frac{P'(z)}{P(z)} \right|,
\]

and we have proved

**Theorem 2.** If \( z \) is a point from which the convex hull of the zeros of the polynomial \( P(z) \) of degree \( n \) subtends an angle \( 2\psi < \pi \), then

(5) \( |P'(z)| \geq n |a_n|^{1/n} (\cos \psi) \left| \frac{P(z)}{P(z)} \right|^{1-(1/n)} \).

**Corollary 1.** The zeros of \( P'(z) \) lie in the convex hull of the zeros of \( P(z) \) (Gauss-Lucas).

**Corollary 2.** If the zeros of \( P(z) \) lie in the unit circle, then we have for \( |z| > 1 \),

(6) \( \left| P'(z) \right| \geq \frac{n |a_n|^{1/n}}{\sqrt{\left( 1 - \frac{1}{|z|^2} \right)}} \left| P(z) \right|^{1-(1/n)} \).
Theorem 3. The zeros of the polynomial

\[ P(z) = a_0 + a_1 z + \cdots + a_n z^n, \]

are not contained in any sector of central angle less than

\[ 2 \cos^{-1} \left\{ \min_{0 \leq k \leq n-1} \left| \frac{a_{n-k}}{a_k} \left( \frac{n}{k} \right)^{1/n-k} \right| \right\}. \]

Proof. Suppose the zeros of \( P(z) \) lie in a sector of angle \( 2\psi < \pi \). From Theorem 1,

\[ \left| \frac{a_0}{a_n} \right|^{1/n} \leq \frac{\sec \psi}{n} \left| \frac{a_{n-1}}{a_n} \right|, \]

or

\[ \sec \psi \geq n \left| a_n \right|^{1-(1/n)} \left| a_0 \right|^{1/n} \left| a_{n-1} \right|^{-1}. \]

Applying this result to

\[ P^{(k)}(z) = \sum_{r=0}^{n-k} \frac{(n+k)!}{\nu!} a_{r+k} z^r, \]

which, by Corollary 1 also satisfies the hypotheses, we find

\[ \sec \psi \geq \left| \frac{na_n}{a_{n-1}} \right|^{1/n-k} \left( \frac{a_n}{a_{n-k}} \left( \frac{n}{k} \right)^{1/(n-k)} \right)^{-1} \]

\( (k = 0, 1, \cdots, n-1) \),

and the result follows.

Theorem 4. Under the hypotheses of Theorem 2, let \( \rho \) denote the distance from \( z \) to the center of gravity of the zeros of \( P(z) \). Then

\[ (7) \quad \left| P(z) \right| \leq \left| a_n \right| (\rho \sec \psi)^n. \]

Proof. Apply Theorem 1 to the numbers \( z - z_1, \cdots, z - z_n \).

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