The Editor's Corner: *n* Coins in a Fountain

ANDREW M. ODLYZKO* AND HERBERT S. WILF**

In Richard Guy's article in last month's Monthly **[6]** there appeared a number of elegant questions, one of which we will answer here.

An (n, k) *fountain* is an arrangement of n coins in rows such that there are exactly k coins in the bottom row, and such that each coin in a higher row touches exactly two coins in the next lower row. In FIG. 1 below we show a (28, 12) fountain. Let $f(n, k)$ be the number of (n, k) fountains, and let $f(n) = \sum_k f(n, k)$ be the number of fountains of n coins.

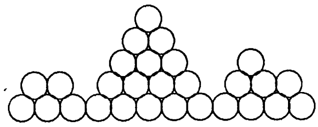

FIG. 1. A (28, 12) fountain.

The question of determining $\{f(n)\}$ was raised in [6], and attributed to J. G. Propp. The values of $f(n)$ for $1 \leqslant n \leqslant 18$ are 1, 1, 2, 3, 5, 9, 15, 26, 45, 78, 135, 234, 406, 704, 1222, 2120, 3679, 6385.

We will show that $\{f(n)\}$ has the remarkable generating function

$$F(x) = 1 + \sum_{n \geqslant 1} f(n) x^n = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x^2}{1 - \cfrac{x^3}{\ddots}}}}. \tag{1}$$

This continued fraction was first studied by Ramanujan (see **[1]**, **[2]**, **[4]**, **[7]**). That the power series coefficients of this fraction have a combinatorial meaning is not new. Already in 1968 Szekeres **[8]** found such an interpretation and others were found later by Flajolet **[5]**. However the interpretation as the number of n-fountains seems particularly attractive, and it lends itself to a very transparent proof.

We say that an (n, k) fountain is *primitive* if its next-to-bottom row contains no empty positions (i.e., contains $k - 1$ coins), and let $g(n, k)$ be the number of primitive (n, k) fountains. By peeling off the bottom row of a primitive fountain we see that

$$f(n - k, k - 1) = g(n, k) \qquad (n \geqslant k;\ k \geqslant 1). \tag{2}$$

Next, let $\mathscr{F}$ be an arbitrary (n, k) fountain, and suppose that the first empty position in the next-to-bottom row is the rth $(1 \leqslant r \leqslant k - 1)$ (in FIG. 2 we have

*AT & T Bell Laboratories, Murray Hill, NJ 07974.

**Supported in part by the United States Office of Naval Research.

$r = 3$). Then by drawing a dotted line to the right of the rth coin in the bottom row, as in Fig. 2, we split $\mathscr{F}$ into an (n', r) primitive fountain and an $(n - n', k - r)$ not-necessarily-primitive fountain.

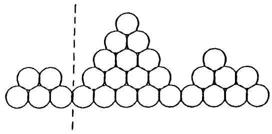

FIG. 2. Split off a primitive fountain.

The factorization is unique, and so we have

$$f(n, k) = \sum_{n', r \geq 0} g(n', r) f(n - n', k - r) \tag{3}$$

for all $n, k \geq 1$, where $f(0,0) = 1$ and $g(0,0) = 0$.

Hence if $F(x, y)$ and $G(x, y)$ are the ordinary generating functions of the sequences f, g respectively, then from (3) we find that $F = 1 + FG$ and from (2) that $G(x, y) = xyF(x, xy)$, whence

$$\begin{aligned} F(x, y) &= \frac{1}{1 - xyF(x, xy)} \\ &= \cfrac{1}{1 - \cfrac{xy}{1 - x^2yF(x, x^2y)}} \\ &= \cdots \\ &= \cfrac{1}{1 - \cfrac{xy}{1 - \cfrac{x^2y}{1 - \cfrac{x^3y}{\ddots}}}}. \end{aligned} \tag{4}$$

In particular, if we put $y = 1$ we obtain the generating function for the numbers of n-fountains in the form (1), as claimed.

Remarks

1. An easier problem would have resulted if we had asked for $h(k) = \sum_n f(n, k)$, the number of fountains whose bottom row contains k coins, without regard to the total number of coins. If we put $x = 1$ in (4) we find that $\{h(k)\}$ is generated by $\mathrm{H}(y)$ where $\mathrm{H}(y) - y\mathrm{H}^2(y) = 1$. Hence the $\{h(k)\}$ are the Catalan numbers.

2. The continued fraction (1) of Ramanujan can be rewritten in various forms, for instance as

$$F(x) = \frac{\displaystyle\sum_{r \geq 0} (-1)^r \frac{x^{r(r+1)}}{(1 - x)(1 - x^2) \cdots (1 - x^r)}}{\displaystyle\sum_{r \geq 0} (-1)^r \frac{x^{r^2}}{(1 - x)(1 - x^2) \cdots (1 - x^r)}}. \tag{5}$$

We will now use this expression for $F(x)$ to deduce the asymptotics of $f(n)$. More precisely, we will show that

$$f(n) = cx_0^{-n} + O\left((5/3)^n\right) \qquad (n \to \infty) \tag{6}$$

where $c = 0.312363324596741\ldots$ and $x_0 = 0.576148769142756\ldots$.

Let $p(x)$ and $q(x)$ denote, respectively, the numerator and the denominator on the right side of (5), and let $p_n(x)$, $q_n(x)$ denote the nth partial sums of the series for $p(x)$ and $q(x)$. It is clear that p and q are analytic in $|x| < 1$, so F is meromorphic there. We will show that $q(x)$ has a simple, real zero $x_0 \in (0.57, 0.58)$, and no other zeros in the disk $|x| < 0.62$, while $p(x_0) > 0$. It will then follow that (6) holds, and more precise numerical calculations yield the more accurate values of x_0 and $c = -p(x_0)/(x_0 q'(x_0))$ stated above.

The proof of our claim about x_0 can be obtained fairly easily using numerical methods to compute multiplicities of zeros of analytic functions. Here we sketch a proof that can be carried out by hand.

Write $q_3(x) = a(x)/((1-x)(1-x^2)(1-x^3))$ where

$$a(x) = 1 - 2x - x^2 + x^3 + 3x^4 + x^5 - 2x^6 - x^7 - x^9,$$

and consider $b(x) = \prod_{j=1}^{9}(x - x_j)$, where the x_j are 0.57577, $-0.46997 \pm i0.81792$, $0.74833 \pm i0.07523$, $-1.05926 \pm i0.36718$, $0.49301 \pm i1.58185$, in that order (these x_j's are approximations to the zeros of $a(x)$). An easy hand calculation shows that if $\{a_k\}$ and $\{b_k\}$ are the coefficients of $a(x)$ and $b(x)$ respectively, then $\sum|a_k - b_k| \leqslant 1.7 \times 10^{-4}$, and so $|a(x) - b(x)| \leqslant 1.7 \times 10^{-4}$ for all $|x| \leqslant 1$. Another such calculation shows that $|b(x)| \geqslant 8 \times 10^{-4}$ for all $|x| = 0.62$.

On the other hand, for $u \in (0, 0.62)$ and $|x| = u$ we have

$$\left|\frac{x^{(k+1)^2 - k^2}}{1 - x^{k+1}}\right| \leqslant \frac{u^{2k+1}}{1 - u^{k+1}} \leqslant \frac{u^9}{1 - u^5} \qquad (k \geqslant 5)$$

and, therefore,

$$\left|\sum_{k=4}^{\infty}(-1)^k \frac{x^{k^2}}{\prod_{j=4}^{k}(1 - x^j)}\right| \leqslant \frac{u^{16}}{1 - u^4}\sum_{m \geqslant 0}\left(\frac{u^9}{1 - u^5}\right)^m \leqslant 6 \times 10^{-4}.$$

Therefore, by Rouché's theorem, $q(x)$ and $b(x)$ have the same number of zeros in $|x| \leqslant 0.62$, namely, 1. Further, since q has real coefficients, its zero is real, and we call it x_0. A brief calculation shows that $x_0 \in (0.57, 0.58)$.

Finally, we must show that $p(x_0) > 0$. However, since the successive summands in the definition of $p(x)$ decrease in absolute magnitude for each fixed real x, $0 < x < 0.6$, we have $p(x) > 1 - x^2/(1 - x)$, which is positive for $0 < x < 0.6$, and the claim is established.

We note that the asymptotic expansion that we have proved is very accurate, even for moderately small values of n. For instance, $f(120) = 1.700213368\ldots \times 10^{28}$, while $f(120) - cx_0^{-120} = 1.59\ldots \times 10^9$.

3. To make the correspondence between our problem and the one that Szekeres solved quite explicit, we draw the slant lines in Fig. 1 that are shown in FIG. 3, and let x_i denote the number of coins on the ith slant line. If we let $q_i = x_i + i - 1$ ($i = 1, k$) then we see that $f(n, k)$ counts partitions with a fixed 'rate of climb.'

Precisely, $f(n, k)$ is the number of partitions of the integer $n + \binom{k}{2}$, whose largest part is k and whose ith-from-smallest part is $\geqslant i$ for each $i = 1, k$, and that is exactly the problem that Szekeres treated in [8].

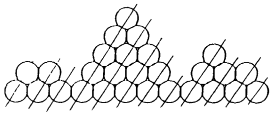

FIG. 3. Draw the slant lines.

4. A problem that is similar to this one has been considered by Auluck [3]. He dealt with arrangements of coins that satisfy our conditions and are further subject to the condition that the set of coins in each horizontal row forms a single contiguous block.

REFERENCES

1. George E. Andrews, An introduction to Ramanujan's 'lost notebook,' this MONTHLY, 86 (1979) 89–108.
2. ______, The theory of partitions, Encycl. Math. and its Appl., vol. 2, Addison-Wesley, 1976.
3. F. C. Auluck, On some new types of partitions associated with generalized Ferrers graphs, *Proc. Cambr. Phil. Soc.*, 47 (1951) 679–686.
4. S. Bhargava and Chandrasekhar Adiga, On some continued fraction identities of Srinavasa Ramanujan, *Proc. Amer. Math. Soc.*, 92 (1984) 13–18.
5. P. Flajolet, Combinatorial aspects of continued fractions, *Discrete Mathematics*, 32 (1980) 125–161.
6. Richard Guy, The strong law of small numbers, this MONTHLY (October, 1988).
7. M. D. Hirschhorn, A continued fraction of Ramanujan, *J. Austral. Math. Soc.* (Ser. A) 29 (1980) 80–86.
8. G. Szekeres, A combinatorial interpretation of Ramanujan's continued fraction, *Canadian Math. Bull.*, 11 (1968) 405–408.