THE EIGENVALUES OF A GRAPH AND ITS CHROMATIC NUMBER

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Let $G$ be a finite, connected, undirected graph, without loops or multiple edges. If $v$ is a vertex of $G$, the degree of $v$, $\rho(v)$, is the number of edges emanating from $v$. R. L. Brooks has shown [1] that

$$k \leq 1 + \max_{v \in G} \rho(v) \tag{1}$$

where $k$ is the chromatic number of $G$, with equality if and only if $G$ is a complete graph or an odd circuit. The estimator (1) may be crude if $G$ has just a few vertices of high degree. An extreme case is the star graph on $n$ vertices

![Star Graph](image)

for which $k = 2$ and (1) gives only $k \leq n$. It seems, therefore, desirable to find an upper estimate, of the character of (1), which is more global in nature, and therefore is less sensitive to the idiosyncrasies of a few uninfluential vertices.

With $G$ we associate the $n \times n$ vertex-adjacency matrix $A = A[G]$, whose $i, j$ entry is 1 if vertices $i$ and $j$ are connected and 0 otherwise. Let $\lambda = \lambda[G] = \lambda_{\text{max}}(A)$ denote the largest eigenvalue of $A$.

**Theorem.** We have

$$k \leq 1 + \lambda \tag{3}$$

with equality if and only if $G$ is a complete graph or an odd circuit.

**Remark.** By the Perron–Frobenius theorem, $\lambda = \max_{v \in G} \rho(v)$, always, so (3) is never inferior to (1). For the graph (2), (3) gives $k = O(\sqrt{n})$.

**Proof of the theorem.** Let the chromatic number of $G$ be $k$. It may be that we can remove a vertex and all edges incident to that vertex from $G$ without lowering the chromatic number. We do this repeatedly, if possible, until a critical graph [2] results, i.e., a graph such that the removal of any star lowers the chromatic number. Let this critical graph be called $G_c$, and suppose it has $m \leq n$ vertices. Consider the following three matrices: $A[G_c]$, the $m \times m$ adjacency matrix of $G_c$; $A'$, the $n \times n$ matrix obtained

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from $A[G]$ by replacing the deleted-vertex rows and columns by zeros; $A[G]$ itself. We have

$$\lambda(G_c) = \lambda_{\text{max}}(A') \leq \lambda(G) = \lambda$$  \hspace{1cm} (4)

the first equality being obvious, and the inequality following from the entry-by-entry domination of $A[G]$ over $A'$.

On the other hand, it is well-known (and indeed, clear) that in a $k$-chromatic critical graph the degree of each vertex is at least $k-1$, and by well-known results about matrices with non-negative elements,

$$\lambda(G_c) \geq k-1$$  \hspace{1cm} (5)

since the smallest row sum in $A[G_c]$ is $\geq k-1$, proving (3).

Now suppose $k=1+\lambda$. Then equality holds in (5), so all the row sums of $G[G_c]$ are equal to $k-1$. Suppose $k>2$. By the theorem of Brooks referred to above, we have equality in (1), and so $G_c$ is a complete graph on $k$ vertices. Hence, after renumbering the vertices, if necessary, $A[G]$ can be brought into the form of an $n \times n$ matrix whose upper left $k \times k$ block is

$$\begin{pmatrix}
0 & 1 & 1 & 1 & \ldots & 1 \\
1 & 0 & 1 & 1 & \ldots & 1 \\
1 & 1 & 0 & 1 & \ldots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \ldots & 0
\end{pmatrix}$$

Consider the $n$-vector $x=(1, 1, \ldots, 1, \epsilon, 0, 0, \ldots, 0)$ whose $(k-1)$-st component is $\epsilon > 0$. Then

$$\lambda \geq \frac{(x, A[G]x)}{(x, x)} \geq \frac{k(k-1)-2\epsilon \sum_{j=1}^{k} a_{j, k+1} + O(\epsilon^2)}{k-\epsilon^2}$$

which is $> k-1$, a contradiction, unless $a_{j, k+1} = 0$ ($j=1, \ldots, k$). Moving the $\epsilon$ to a different position in $x$, we conclude that $a_{j, r} = 0$ ($j=1, \ldots, k$; $r=k+1, \ldots, n$), hence $G$ is disconnected, a contradiction, and so $n=k$. The case $k=2$ can be handled similarly.

**Corollary.** Let $G$ have $E$ edges and $n$ vertices. Then

$$k \leq \left(2 \left(1 - \frac{1}{n}\right) E\right)^{\frac{1}{4}} + 1$$

with equality only for complete graphs.

**Proof.** If $\Sigma \lambda_i = 0$, then

$$\max_i \lambda_i \leq \left(1 - \frac{1}{n} \sum_{i=1}^{n} \lambda_i^2\right)^{\frac{1}{2}}$$
hence

\[ \lambda_{\text{max}}(A) \leq \left( \left( 1 - \frac{1}{n} \right) \text{Trace}(A^2) \right)^{\frac{1}{2}} \]

\[ = \left( 2 \left( 1 - \frac{1}{n} \right) E \right)^{\frac{1}{4}}. \]

References


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