Closed form summation of C-finite sequences*

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Dedicated to David P. Robbins

Abstract

Suppose $\{F(n)\}$ is a sequence that satisfies a recurrence with constant coefficients whose associated polynomial equation has distinct roots. Consider a sum of the form

$$\sum_{j=0}^{n-1}(F(a_1n+b_1j+c_1)F(a_2n+b_2j+c_2)\dots F(a_kn+b_kj+c_k)).$$

We prove that such a sum always has a closed form in the sense that it evaluates to a polynomial with a fixed number of terms, in the values of the sequence $\{F(n)\}$. We explicitly describe two different sets of monomials that will form such a polynomial, and give an algorithm for finding these closed forms, thereby completely automating the solution of this class of summation problems. We exhibit tools for determining when these explicit evaluations are unique of their type, and prove that in a number of interesting cases they are indeed unique.

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1 Introduction

In section 1.6 of [5] the following assertion is made:

> All Fibonacci number identities such as Cassini's $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ (and much more complicated ones), are routinely provable using Binet's formula:
>
> $$F_n := \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

This is followed by a brief Maple program that proves Cassini's identity by substituting Binet's formula on the left side and showing that it then reduces to $(-1)^n$. Another method of proving these identities is given in [7], in which it is observed that one can find the recurrence relations that are satisfied by each of the two sides of the identity in question, show that they are the same and that the initial values agree, and the identity will then be proved.

The purpose of this note is to elaborate on these ideas by showing how to *derive*, instead of only to *verify*, summation identities for a certain class of sequence sums, and to show that this class of sums always has closed form in a certain sense. The procedure for evaluating these sums in closed form is thereby entirely automated.

We deal with the class of C-finite sequences (see [7]). These are the sequences $\{F(n)\}_{n\geq 0}$ that satisfy linear recurrences of fixed span with constant coefficients. We suppose further that the roots of the associated polynomial equation are distinct. The Fibonacci numbers, e.g., will do nicely for a prototype sequence of this kind.

The kind of sum that we will deal with will be of the form (2) below. We will say that such a sum has an *F-closed form* if there is a linear combination of a fixed (i.e., independent of n) number of monomials in values of the F's such that for all n the sum $f(n)$ is equal to that linear combination.

For example, look at the sum (9) in Section 3.1, where the F's are the Fibonacci numbers. The evaluation (10) of that sum as a linear combination of five monomials in the F's shows that the sum has F-closed form.

Suppose that our C-finite sequence $\{F(n)\}_{n\geq 0}$ has the form

$$F(n) = \sum_{m=1}^{d} \lambda_m r_m^n, \tag{1}$$

in which the r_m's are distinct and nonzero. We are interested in evaluating a sum

$$f(n) = \sum_{j=0}^{n-1} \left(F(a_1 n + b_1 j + c_1) \cdots F(a_k n + b_k j + c_k)\right) \tag{2}$$

in which the a's, b's, and c's are given integers. It is elementary and well known that $f(n)$ is C-finite, and one can readily obtain explicit expressions for $f(n)$ in terms of the roots r_i. Our main results show how to obtain formulæ for $f(n)$ as a polynomial in the F's, based on two different explicit sets of "target" monomials in the F's. Using the first target set, we get the following result.

Theorem 1 *The sum $f(n)$ in (2) has an F-closed form. It is in fact equal to a linear combination of at most $3d^k$ monomials in the F's, namely,*

$$\begin{array}{ll} F(a_1n+i_1)F(a_2n+i_2)\dots F(a_kn+i_k), & 0 \le i_\nu \le d-1, \\ nF(a_1n+i_1)F(a_2n+i_2)\dots F(a_kn+i_k), & 0 \le i_\nu \le d-1, \\ F((a_1+b_1)n+i_1)\dots F((a_k+b_k)n+i_k), & 0 \le i_\nu \le d-1. \end{array} \tag{3}$$

The coefficients in this linear combination can be found by equating at most $3d^k$ values of the sum $f(n)$ to the values of the assumed linear combination, and solving for the coefficients. If F is rational-valued then there are rational solutions.

The second target set gives an alternate F-closed form.

Theorem 2 *The sum $f(n)$ in (2) can be expressed in F-closed form as a linear combination of monomials of the form*

$$\begin{array}{ll} F(n+i_1)F(n+i_2)\dots F(n+i_Q), & 0 \le i_\nu \le d-1, \\ nF(n+i_1)F(n+i_2)\dots F(n+i_Q), & 0 \le i_\nu \le d-1, \\ F(n+i_1)F(n+i_2)\dots F(n+i_P), & 0 \le i_\nu \le d-1, \end{array} \tag{4}$$

where $Q = a_1+a_2+\cdots+a_k$ and $P = (a_1+b_1)+(a_2+b_2)+\cdots+(a_k+b_k)$. The coefficients can be found by solving equations involving at most $2d^Q + d^P$ values of $f(n)$. If F is rational-valued then there are rational solutions.

When $F(n)$ is a Fibonacci number, Theorem 2 states that any sum of monomials in the F's can be rationally expressed as a linear combination of monomials in $F(n)$ and $F(n+1)$, where these monomials have at most two different degrees.

The natural domain for these questions is the vector space $\mathbf{V}^\infty$ of complex-valued functions on $\{0, 1, 2, \dots\}$. However, our expansions can be obtained by solving linear equations in the vector space $\mathbf{V}^N$ of complex-valued functions on $\{1, \dots, N\}$ for various values of N. This is justified by the following lemma.

Lemma 3 *Let $\mathbf{W}^N$ be the vector space of complex-valued functions on $\{1, 2, \ldots, N\}$ spanned by the monomials in (3), where $N = 3d^k$, and let $\mathbf{W}^\infty$ be the vector space of functions on $\{0, 1, \ldots, \}$ spanned by the same monomials. If two linear combinations of monomials of type (3) agree in $\mathbf{W}^N$, then they agree in $\mathbf{W}^\infty$. A similar statement holds for monomials of type (4).*

In general, F-closed expressions are not unique. For example, we may add terms of the form $\Psi(F)(F(n+2) - F(n+1) - F(n))$, where $\Psi(F)$ is any polynomial in the $F(an+i)$, to an expression involving Fibonacci numbers and get another valid F-closed form. However, the formats described by (3) and (4) turn out to be highly restrictive, and the resulting expressions can be shown to be unique in a surprising number of cases. We will return to the question of uniqueness and, more generally, the problem of computing $\dim(\mathbf{W}^\infty)$, in Sections 4 and 5.

2 Proofs

If we expand the right side of (2) above, using (1), we find that

$$f(n) = \sum_{j=0}^{n-1} \prod_{\ell=1}^{k} \left\{ \sum_{m=1}^{d} \lambda_m r_m^{a_\ell n + b_\ell j + c_\ell} \right\}.$$

A typical term in the expansion of the product will look like

$$K r_{m_1}^{a_1 n + b_1 j + c_1} r_{m_2}^{a_2 n + b_2 j + c_2} \ldots r_{m_k}^{a_k n + b_k j + c_k}, \tag{5}$$

in which K is a constant, i.e., is independent of n and j, which may be different at different places in the exposition below. Since we are about to sum the above over $j = 0 \ldots n-1$, put

$$\Theta = r_{m_1}^{b_1} r_{m_2}^{b_2} \ldots r_{m_k}^{b_k},$$

because this is the quantity that is raised to the jth power in the expression (5). Now there are two cases, namely $\Theta = 1$ and $\Theta \neq 1$.

Suppose $\Theta = 1$. Then the sum of our typical term (5) over $j = 0 \ldots n-1$ is

$$Kn \left(r_{m_1}^{a_1} r_{m_2}^{a_2} \ldots r_{m_k}^{a_k} \right)^n. \tag{6}$$

On the other hand, if $\Theta \neq 1$ then the sum of our typical term (5) over $j = 0 \ldots n-1$ is[1]

$$K \left\{ \left(r_{m_1}^{a_1 + b_1} \ldots r_{m_k}^{a_k + b_k} \right)^n - \left(r_{m_1}^{a_1} \ldots r_{m_k}^{a_k} \right)^n \right\}. \tag{7}$$

[1] Eqs. (6),(7) show clearly that our sum of C-finite functions is itself C-finite.

The next task will be to express these results in terms of various members of the sequence $\{F(n)\}$ instead of in terms of various powers of the r_i's. To do that we write out (1) for d consecutive values of n, getting

$$\begin{aligned} F(n+i) &= \sum_{m=1}^{d} \lambda_m r_m^{n+i} \qquad (i = 0, 1, \ldots, d-1) \\ &= \sum_{m=1}^{d} (\lambda_m r_m^i) r_m^n \qquad (i = 0, 1, \ldots, d-1). \end{aligned}$$

We regard these as d simultaneous linear equations in the unknowns $\{r_1^n, \ldots, r_d^n\}$, with a coefficient matrix that is a nonsingular diagonal matrix times a Vandermonde based on distinct points, and is therefore nonsingular. Hence *for each* $m = 1, \ldots, d$, r_m^n *is a linear combination of* $F(n), F(n+1), \ldots, F(n+d-1)$*, with coefficients that are independent of* n. Thus in eqs. (6), (7) we can replace each $r_{m_i}^{na_i}$ by a linear combination of $F(a_i n), F(a_i n + 1), \ldots, F(a_i n + d - 1)$ and we can replace each $r_{m_i}^{n(a_i+b_i)}$ by a linear combination of $F((a_i + b_i)n), F((a_i + b_i)n + 1), \ldots, F((a_i + b_i)n + d - 1)$.

After making these replacements, we see that the two possible expressions (6), (7) contribute monomials that are all of the form (3). This establishes the existence of expansions in monomials of type (3), as claimed in Theorem 1.

To prove the corresponding claim made in Theorem 2, it suffices to observe that, in the above argument, we could have written $r_{m_i}^{na_i} = (r_{m_i}^n)^{a_i}$ and replaced it by a homogeneous polynomial of degree a_i in $F(n), F(n+1), \ldots, F(n+d-1)$. Similar reasoning applies to $r_{m_i}^{n(a_i+b_i)}$. Thus all of the resulting monomials are of type (4).

To complete the proofs of Theorems 1 and 2, it remains to show that the coefficients can be found by solving equations in $\mathbf{V}^N$, where $N = 3d^k$ in case (3) and $N = 2d^Q + d^P$ in case (4). The preceding arguments show that, in each of the cases (3) or (4), if we compute $f(1), f(2), \ldots, f(N)$ and equate these values to a linear combination of monomials with unknown coefficients, there exists at least one solution in $\mathbf{V}^N$. Furthermore, it is clear that, if the values of f are rational, there exist solutions with rational coefficients. It thus remains only to prove Lemma 3, which shows that the solution obtained is also valid in $\mathbf{V}^\infty$, i.e., it agrees with $f(n)$ for all n. The proofs are identical for cases (3) and (4), so we will consider only the former.

We have observed that for each $i = 0, \ldots, d-1$, $F(n+i)$ is a linear combination of $r_1^n, \ldots, r_d^n$, and conversely. Hence, in both $\mathbf{V}^N$ and $\mathbf{V}^\infty$, the linear span of the set

$$F(a_1 n + i_1) F(a_2 n + i_2) \ldots F(a_k n + i_k), \qquad 0 \le i_\nu \le d-1$$

is equal to the linear span of the set $\{\theta_1^n, \theta_2^n, \ldots, \theta_{d^k}^n\}$, where the θ_j range over all monomials of the form

$$r_{m_1}^{a_1} r_{m_2}^{a_2} \cdots r_{m_k}^{a_k}.$$

Similarly, the linear span of all $3d^k$ monomials of type (3) is equal to the linear span of the set of $3d^k$ functions

$$\begin{gathered} \theta_1^n, \theta_2^n, \ldots, \theta_{d^k}^n \\ \psi_1^n, \psi_2^n, \ldots, \psi_{d^k}^n \\ n\psi_1^n, n\psi_2^n, \ldots, n\psi_{d^k}^n, \end{gathered} \tag{8}$$

where the θ_i are as defined above and the ψ_j range over all monomials of the form

$$r_{m_1}^{a_1+b_1} r_{m_2}^{a_2+b_2} \cdots r_{m_k}^{a_k+b_k}.$$

Suppose that $\Phi(n)$ and $\Psi(n)$ are linear combinations of monomials of type (3), with $\Phi(n) = \Psi(n)$ for $n = 1, 2, \ldots, N$. We know that $\Phi(n)$ and $\Psi(n)$ can both be expressed in the form

$$\sum_{i=1}^{d^k} c_i \, \theta_i^n + \sum_{j=1}^{d^k} d_j \, \psi_j^n + \sum_{k=1}^{d^k} e_k \, n\psi_k^n$$

for some constants c_i, d_j, e_k. It follows from standard results in the theory of difference equations (e.g., see [2], Chapter 11) that both $\Phi(n)$ and $\Psi(n)$ satisfy a single linear recurrence of order $3d^k$ with constant coefficients, i.e., the recurrence with characteristic polynomial $\prod_i (t - \theta_i) \prod_j (t - \psi_j)^2$. Hence the values of $\Phi(n)$ and $\Psi(n)$ are completely determined by their values for $n = 1, 2, \ldots, N$, and since they agree for these values, they must agree for all n. This completes the proof of Lemma 3, and hence also of the proofs of Theorem 1 and Theorem 2.

3 Examples

3.1 A Fibonacci sum

This work was started when a colleague asked about the sum

$$f(n) = \sum_{j=0}^{n-1} F(j)^2 F(2n - j), \tag{9}$$

in which the F's are the Fibonacci numbers. If we refer to the general form (2) of the question we see that in this case

$$k = 3; d = 2; (a_1, b_1, c_1) = (a_2, b_2, c_2) = (0, 1, 0); (a_3, b_3, c_3) = (2, -1, 0).$$

If we now refer to the general form (3) of the answer we see that the sum $f(n)$ is a linear combination of monomials

$$nF(2n), F(2n), nF(2n+1), F(2n+1), F(n)^3, F(n)^2F(n+1), F(n)F(n+1)^2, F(n+1)^3.$$

Hence we assume a linear combination of these monomials and equate its values to those of $f(n)$ for $n = 0, 1, \ldots, 7$ to determine the constants of the linear combination. The result is that

$$f(n) = \frac{1}{2}\left(F(2n) + F(n)^2F(n+1) - F(n)F(n+1)^2 + F(n+1)^3 - F(2n+1)\right). \tag{10}$$

This formula is expressed in terms of monomials of type (3). Using the identities

$$F(2n) = 2F(n)F(n+1) - F(n)^2 \quad \text{and} \quad F(2n+1) = F(n+1)^2 + F(n)^2$$

we obtain an alternate expression of type (4), namely

$$\begin{aligned} f(n) \;&=\; \frac{1}{2}\Big(2F(n)F(n+1) - 2F(n)^2 - F(n+1)^2 + F(n)^2F(n+1) \\ &\qquad -F(n)F(n+1)^2 + F(n+1)^3\Big) \end{aligned} \tag{11}$$

In Section 5 we will show that both of these expression are unique, i.e., (10) is the unique F-closed formula for $f(n)$ of type (3) and (11) is the unique F-closed formula of type (4).

3.2 An example involving subword avoidance

Given an alphabet of $A \geq 2$ letters, let W be some fixed word of three letters such that no proper suffix of W is also a proper prefix of W. For example, $W = aab$ will do nicely. Let $G(n)$ be the number of n-letter words over A that do not contain W as a subword. It is well known, and obvious, that

$$G(n) = AG(n-1) - G(n-3), \tag{12}$$

so this is a C-finite sequence, and it is easy to check that the roots of its associated polynomial equation are distinct for all $A \geq 2$. Suppose we want to evaluate the sum $g(n) = \sum_{j=0}^{n-1} G(j)^2$. Then

$$k = 2; d = 3; (a_1, b_1, c_1) = (a_2, b_2, c_2) = (0, 1, 0).$$

Using either Theorem 1 or Theorem 2, we see that $g(n)$ is a linear combination of the monomials

$$1, n, G(n)^2, G(n)G(n+1), G(n)G(n+2), G(n+1)G(n+2), G(n+1)^2, G(n+2)^2.$$

As before, we assume a linear combination of these monomials with constants to be determined, and we equate the result to $g(n)$, for $n = 0, 1, \ldots, 7$, to solve for the constants. The end result is that

$$\begin{aligned} g(n) &= \frac{1}{A(A-2)}\Big(1 - (A-1)^2 G(n)^2 - 2G(n)G(n+1) + 2G(n)G(n+2) \\ &\quad +2(A-1)G(n+1)G(n+2) - (A-1)^2 G(n+1)^2 - G(n+2)^2\Big) \end{aligned} \tag{13}$$

if $A > 2$, and

$$\begin{aligned} g(n) &= n + 2G(n)^2 + 7G(n)G(n+1) - 5G(n)G(n+2) \\ &\quad -5G(n+1)G(n+2) + 3G(n+1)^2 + 2G(n+2)^2 \end{aligned} \tag{14}$$

if $A = 2$.

In the latter case, it is easy to show that $G(n) = F(n) - 1$ where $F(n)$ is a standard Fibonacci number. Consequently, $G(n+2) - G(n+1) - G(n) = 1$, and adding any multiple of the relation

$$(G(n+2) - G(n+1) - G(n) - 1)^2 = 0 \tag{15}$$

to the right side of (14) gives another degree 2 expression of type (3) or (4). Thus formula (14) is not unique within the class of formulæ of type (3) or (4). However, in Section 5 we will show that, when $A > 2$, formula (13) is unique within this class. When $A = 2$, we show that all relations are constant multiples of (15).

3.3 Fibonacci power sums

Theorem 1 and Theorem 2 imply that, if the $F(j)$'s are the Fibonacci numbers then for each integer $p = 1, 2, \ldots$ there is a formula

$$f(n) = \sum_{j=0}^{n-1} F(j)^p = \sum_{j=0}^{p} \Lambda_{p,j} F(n)^j F(n+1)^{p-j} + c_p n + d_p.$$

Here is a brief table of values of these coefficients.

p	$\Lambda_{p,0}$	$\Lambda_{p,1}$	$\Lambda_{p,2}$	$\Lambda_{p,3}$	$\Lambda_{p,4}$	$\Lambda_{p,5}$	$\Lambda_{p,6}$	$\Lambda_{p,7}$	c_p	d_p
1	1	0	0	0	0	0	0	0	0	-1
2	0	1	-1	0	0	0	0	0	0	0
3	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{3}{2}$	0	0	0	0	0	$\frac{1}{2}$
4	0	$\frac{2}{25}$	$-\frac{3}{25}$	$\frac{14}{25}$	$-\frac{19}{25}$	0	0	0	$\frac{6}{25}$	0
5	$\frac{7}{22}$	$-\frac{5}{22}$	$-\frac{15}{11}$	$\frac{10}{11}$	$\frac{15}{11}$	$-\frac{15}{22}$	0	0	0	$-\frac{7}{22}$
6	0	$\frac{1}{2}$	$-\frac{5}{4}$	0	$\frac{5}{4}$	$\frac{1}{2}$	-1	0	0	0
7	$-\frac{139}{638}$	$\frac{763}{638}$	$-\frac{945}{638}$	$-\frac{350}{319}$	$\frac{105}{58}$	$\frac{357}{319}$	$-\frac{105}{319}$	$-\frac{777}{638}$	0	$\frac{139}{638}$

The resulting expressions for $f(n)$ turn out to be unique within the class of type (3) or (4) formulæ when $p \not\equiv 0 \bmod 4$. When p is a multiple of 4 (for example, in the fourth line of the table above) the formulæ are not unique, but are subject to a one-parameter family of relations generated by powers of the degree-4 relation

$$(F(n+1)^2 - F(n)^2 - F(n)F(n+1))^2 = 1 \quad .$$

We will establish these facts in Section 5.

3.4 Generic power sums

Consider the power sum

$$f(n) = \sum_{k=0}^{n-1} F(k)^2,$$

where F is a "generic" C-finite function of order 2. In other words, $F(n)$ solves a linear recurrence

$$F(n) = AF(n-1) + BF(n-2)$$

with initial values $F(0)$ and $F(1)$, where A and B are independent transcendentals. In fact, for this example we only need to assume that, if r_1 and r_2 are the associated roots, then r_1

and r_2 are distinct and none of the monomials $r_1^2, r_1 r_2$, and r_2^2 equals 1. This is equivalent to assuming simply that $A^2 + 4B \neq 0$, $A \neq \pm(B-1)$, and $B \neq -1$.

Using techniques introduced earlier, we can express $f(n)$ as a linear combination of $F(n)^2, F(n)F(n+1), F(n+1)^2$, and 1. The solution may be computed explicitly in terms of $A, B, F(0)$, and $F(1)$, and we find that $f(n)$ equals

$$\frac{(1-B-A^2(1+B))F(n)^2 + (2AB)F(n)F(n+1) + (1-B)F(n+1)^2 - K}{(A^2-(B-1)^2)(B+1)} \tag{16}$$

where

$$K = (1-B-A^2(B+1)1))F(0)^2 + (2AB)F(0)F(1) + (1-B)F(1)^2.$$

In (16), we observe a curious phenomenon: since $F(n)$ depends on $F(0)$ and $F(1)$, we might expect that our linear equations would have led to a solution in which each of the coefficients depends on $F(0)$ and $F(1)$. However, this dependence appears only in the constant term. The next theorem demonstrates that such behavior is typical for power sums of C-finite functions in which the terms in (4) containing n are not present, i.e, in cases where no monomial in the roots equals 1.

Theorem 4 *Suppose that $\{F(n)\}_{n\geq 0}$ is a C-finite sequence determined by a recurrence of order d together with initial conditions $F(0), F(1), \ldots, F(d-1)$. Suppose that the recurrence polynomial has distinct roots $r_1, \ldots, r_d$, and suppose that no monomial of degree p in the r_i equals 1. Let $f(n) = \sum_{j=0}^{n-1} F(j)^p$, where p is a positive integer, and let*

$$f(n) = \sum_{0 \leq i_1, i_2, \ldots, i_d \leq d-1} \Lambda_{i_1, i_2, \ldots, i_d} F(n+i_1)F(n+i_2)\cdots F(n+i_d) \ + \ K \tag{17}$$

be the expansion of $f(n)$ obtained according to the method given in Section 2. Then the coefficients $\Lambda_{i_1,i_2,\ldots,i_d}$ in (17) do not depend on $F(0), F(1), \ldots, F(d-1)$.

Proof: Suppose that $F(n) = \sum_{m=1}^{d} \lambda_m r_m^n$. Define

$$\mathbf{X}(n) = \begin{pmatrix} \lambda_1 r_1^n \\ \lambda_2 r_2^n \\ \vdots \\ \lambda_d r_d^n \end{pmatrix} \quad \text{and} \quad \mathbf{Y}(n) = \begin{pmatrix} F(n) \\ F(n+1) \\ \vdots \\ F(n+d-1) \end{pmatrix}.$$

Then we have

$$\mathbf{Y}(n) \ = \ \mathbf{V}\,\mathbf{X}(n) \quad \text{and} \quad \mathbf{X}(n) = \mathbf{V}^{-1}\mathbf{Y}(n) \tag{18}$$

where $\mathbf{V}$ is a Vandermonde matrix in the r_i. It follows from (18) that the terms $\lambda_m r_m^n, 1 \leq m \leq d$, can be expressed as linear combinations of the functions $F(n+i)$, with coefficients that do not depend on $F(0), F(1), \ldots, F(d-1)$. Using the method of Section 2, we can compute

$$
\begin{aligned}
f(n) = \sum_{j=0}^{n-1} F(j)^p &= \sum_{j=0}^{n-1} \Big(\sum_{m=0}^{d} \lambda_m r_m^j \Big)^p \\
&= \sum_{j=0}^{n-1} \Big(\sum_{0 \le i_1, i_2, \ldots, i_p \le d} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_p} (r_{i_1} r_{i_2} \cdots r_{i_p})^j \Big) \\
&= \sum_{0 \le i_1, i_2, \ldots, i_p \le d} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_p} \frac{(r_{i_1} r_{i_2} \cdots r_{i_p})^n - 1}{(r_{i_1} r_{i_2} \cdots r_{i_p}) - 1} \\
&= \sum_{0 \le i_1, i_2, \ldots, i_p \le d} \frac{(\lambda_{i_1} r_{i_1}^n)(\lambda_{i_2} r_{i_2}^n) \cdots (\lambda_{i_p} r_{i_p}^n)}{(r_{i_1} r_{i_2} \cdots r_{i_p}) - 1} - K \qquad (19)
\end{aligned}
$$

where K is a constant. Using (18), we can express all of the terms in (19) except K as a linear combination of monomials in the $F(n+i)$ with coefficients that do not depend on $F(0), F(1), \ldots, F(d-1)$, as claimed.

4 Uniqueness: Fibonacci power sums

Motivated by Examples 3.3 and 3.4, we consider the question of whether expansions of the form

$$\sum_{j=0}^{p} \Lambda_{p,j} F(n)^j F(n+1)^{p-j}$$

and, more generally,

$$\sum_{j=0}^{p} \Lambda_{p,j} F(n)^j F(n+1)^{p-j} + c_p n + d_p.$$

are unique. Here, $F(n)$ denotes the nth Fibonacci number. In Section 5 we develop general tools to help answer these questions for solutions of more general linear recurrences and other summations such as those arising in Examples 3.1 and 3.2. The techniques in these two sections can be viewed as refinements and extensions of the ideas introduce in Section 2 to prove Theorem 1, Theorem 2, and Lemma 3.

Theorem 5 *Let* $\mathbf{V} = \mathbf{V}^\infty$ *denote the vector space of complex-valued functions on* $\{0, 1, 2, \ldots\}$, *and let* $\mathbf{W_p}$ *denote the subspace of* $\mathbf{V}$ *spanned by functions of the form* $F(n)^i F(n+1)^{p-i}$ *for* $i = 0, \ldots, p$, *and let* $\mathbf{W_p^{++}}$ *denote the subspace spanned by the same monomial expressions together with with the functions* $g(n) = n$ *and* $h(n) = 1$. *Then*

(a) $\dim(\mathbf{W_p}) = p+1$

(b) $\dim(\mathbf{W_p^{++}}) = \begin{cases} p+2 & \textit{if } p \textit{ is divisible by } 4 \\ p+3 & \textit{otherwise} \end{cases}$

Corollary 6 *The functions $F(n)^i F(n+1)^{p-i}, 1 \leq i \leq p$ are linearly independent and the set $\{F(n)^i F(n+1)^{p-i}\}_{1\leq i \leq p} \cup \{n, 1\}$ is linearly independent unless p is divisible by 4, in which case there is a single relation among its elements.*

Proof: Let $r_1 = (1+\sqrt{5})/2$ and $r_2 = (1-\sqrt{5})/2$ denote the roots of the Fibonacci recurrence polynomial. As noted earlier the proof of Theorem 1, r_1^n and r_2^n may be expressed as linear combinations of $F(n)$ and $F(n+1)$ and vice versa. Consequently, $\mathbf{W_p}$ is the linear span of $r_1^{ni} r_2^{n(p-i)}, i = 0, \ldots, p$, and to prove statement (a) it suffices to show that these functions are linearly independent. But this follows immediately from the fact that the numbers $r_1^i r_2^{(p-i)}$ are distinct, for $i = 0, \ldots, p$.

To prove part (b), consider the $(p+3) \times (p+3)$ matrix M_p whose first $p+1$ columns are the vectors $(1, \theta_i, \theta_i^2, \ldots, \theta_i^{p+2})$, where $\theta_i = r_1^i r_2^{(p-i)}, i = 0, \ldots, p$, and whose last two columns are the vectors $(1, 1, \ldots, 1)$ and $(0, 1, \ldots, p+2)$. For example, when $p = 2$ we have

$$M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ r_1^2 & r_1 r_2 & r_2^2 & 1 & 1 \\ r_1^4 & r_1^2 r_2^2 & r_2^4 & 1 & 2 \\ r_1^6 & r_1^3 r_2^3 & r_2^6 & 1 & 3 \\ r_1^8 & r_1^4 r_2^4 & r_2^8 & 1 & 4 \end{pmatrix}$$

Note that $\det M_p$ is the derivative at $t = 1$ of the $(p+3) \times (p+3)$ Vandermonde determinant $\det M_p(t)$, where $M_p(t)$ is the matrix whose first $p+2$ columns are the same as those of M_p, and whose last column is $(1, t, t^2, \ldots, t^{p+2})$. We have

$$\begin{aligned} \det M_p &= \frac{d}{dt} \det M_p(t) \Big|_{t=1} \\ &= \frac{d}{dt} \Big(\prod_{0\leq i<j\leq p} (\theta_j - \theta_i) \prod_{0\leq i\leq p} (1-\theta_i)\ (t-1) \prod_{0\leq i \leq p} (t-\theta_i) \Big) \Big|_{t=1} \end{aligned}$$

It follows that $\det M_p = 0$ only when $t = 1$ is a multiple root of $\det M_p(t)$, i.e., $r_1^i r_2^{p-i} = 1$ for some i. Using the fact that $r_1 r_2 = -1$, it is easy to show that this property holds if and only if p is a multiple of 4. Thus, when p is not a multiple of 4, the columns of M_p are linearly independent and we have $\dim(\mathbf{W_p^{++}}) = p+3$.

If p is a multiple of 4, then M_p contains exactly two columns of 1s. If one of these columns is suppressed, the argument just given shows that the remaining columns are linearly independent. Hence $\mathrm{rank}(M_p) = p+2$ and $\dim(\mathbf{W_p^{++}}) \geq p+2$. Since the dimension is clearly at most $p+2$ in this case, the theorem is proved.

5 Uniqueness: general case

Analogs of Theorem 5 hold for more general recurrences, but the exact statements depend on properties of the associated roots. The following theorem can be applied to give precise results in many cases.

Theorem 7 *Let $F(n)$ be the solution to a linear recurrence of order d whose associated roots $r_1, r_2, \ldots, r_d$ are distinct, and let p and q be distinct positive integers. Let $\mathbf{W_p}$ denote the subspace of $\mathbf{V} = \mathbf{V}^\infty$ spanned by degree p monomials of the form*

$$F(n)^{i_1} F(n+1)^{i_2} \cdots F(n+d-1)^{i_d}$$

where $i_1 + i_2 + \cdots i_d = p$ and $i_j \geq 0$ for all j. Let $\mathbf{W_q^+}$ denote the subspace spanned by the degree q monomials

$$\begin{gathered} F(n)^{i_1} F(n+1)^{i_2} \cdots F(n+d-1)^{i_d} \quad \textit{and} \\ nF(n)^{i_1} F(n+1)^{i_2} \cdots F(n+d-1)^{i_d} \end{gathered}$$

where $i_1 + i_2 + \cdots i_d = q$ and $i_j \geq 0$ for all j. And, finally, let $\mathbf{W_{p,q}^{++}} = \mathbf{W_p} + \mathbf{W_q^+}$ denote the subspace spanned by all of the above monomials. Then

$$\dim(\mathbf{W_p}) = |S_p|\ , \quad \dim(\mathbf{W_q^+}) = 2|S_q|\ , \quad \textit{and} \quad \dim(\mathbf{W_{p,q}^{++}}) = |S_p| + 2|S_q| - |S_p \cap S_q|$$

where $S_p = \{r_1^{i_1} r_2^{i_2} \cdots r_d^{i_d} \mid i_1 + i_2 + \cdots i_d = p\}$ and $S_q = \{r_1^{i_1} r_2^{i_2} \cdots r_d^{i_d} \mid i_1 + i_2 + \cdots i_d = q\}$ are the sets of monomials in the r_i of degrees p and q, respectively, both viewed as subsets of the complex numbers.

Corollary 8 *The sets of monomials generating $\mathbf{W_p}$, $\mathbf{W_q^+}$, and $\mathbf{W_{p,q}^{++}}$, respectively, are linearly independent if and only if evaluations of formally distinct monomials in the sets S_p, S_q and $S_p \cup S_q$ yield distinct complex numbers.*

The proof is analogous to that given for Theorem 5, but requires a little more technical machinery. First consider the case of $\mathbf{W_p}$. As noted in Section 2, each of the functions $F(n), F(n+1), \ldots, F(n+d-1)$ lies in the linear span of $r_1^n, r_2^n, \ldots, r_d^n$, and conversely.

Hence $\mathbf{W_p}$ is spanned by the set $\{\theta_1^n, \theta_2^n, \ldots, \theta_{m(p,d)}^n\}$, where $m(p,d) = \binom{p+d-1}{d}$ and the θ_j range over the $m(p,d)$ formally distinct monomials of degree p in $r_1, r_2, \ldots, r_d$. It follows that $\dim \mathbf{W_p}$ equals the rank of the $m(p,d) \times m(p,d)$ matrix whose jth column is equal to $(1, \theta_j, \theta_j^2, \ldots, \theta_j^{m(p,d)-1})$. A familiar argument shows that this rank is equal to the number of distinct values of θ_j, proving that $\dim(\mathbf{W_p}) = |S_p|$.

Similar reasoning shows that $\mathbf{W_q^+}$ is spanned by the $2m(q,d)$ functions

$$\begin{array}{c} \psi_1^n, \psi_2^n, \ldots, \psi_{m(q,d)}^n \\ n\psi_1^n, n\psi_2^n, \ldots, n\psi_{m(q,d)}^n \quad , \end{array} \tag{20}$$

where the ψ_j range over all formally distinct monomials of degree q in $r_1, \ldots, r_d$. Finally, $\mathbf{W_{p,q}^{++}}$ is spanned by the $m(p,d) + 2m(q,d)$ functions

$$\begin{array}{c} \theta_1^n, \theta_2^n, \ldots, \theta_{m(p,d)}^n \\ \psi_1^n, \psi_2^n, \ldots, \psi_{m(q,d)}^n \\ n\psi_1^n, n\psi_2^n, \ldots, n\psi_{m(q,d)}^n, \end{array} \tag{21}$$

where θ_i and ψ_j are defined as above.

To complete the computation of $\dim(\mathbf{W_q^+})$ and $\dim(\mathbf{W_{p,q}^{++}})$, it suffices to show that the sets of functions defined by (20) and (21) are linearly independent if and only if, in each case, the corresponding sets of roots $\{\theta_i\}$ and $\{\theta_i\} \cup \{\psi_j\}$ are distinct. One approach is to invoke standard results in the theory of finite difference equations (e.g. [2], Chapter 11). Alternatively, one can give a direct Vandermonde-type argument based on the following determinant formula (see [1], but also [4] for a more extensive history of this elegant result).

Theorem 9 *Let $x_1, x_2, \ldots, x_n$ be indeterminates, and let $a_1, a_2, \ldots, a_n$ be positive integers with $\sum_i a_i = N$. For any t, and for any integer $k \geq 1$, let*

$$\rho_N(t,k) = \frac{d^k}{dt^k}(1, t, t^2, \ldots, t^{N-1})$$

Let $M(a_1, a_2, \ldots, a_n)$ be the $N \times N$ matrix whose first a_1 rows are $\rho_N(x_1, 0), \ldots, \rho_N(x_1, a_1 - 1)$, and whose next a_2 rows are $\rho_N(x_2, 0), \ldots, \rho_N(x_2, a_2 - 1)$, and so forth. Then

$$\det M(a_1, \ldots, a_n) \; = \; \prod_{i=1}^{n} (a_i - 1)!!! \prod_{1 \leq i < j \leq n} (x_j - x_i)^{a_i a_j}$$

where $k!!!$ denotes $1!2! \cdots k!$ and $0!!! = 1$.

For example, if

$$M(1,2,3) = \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 0 & 1 & 2x_2 & 3x_2^2 & 4x_2^3 & 5x_2^4 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \\ 0 & 1 & 2x_3 & 3x_3^2 & 4x_3^3 & 5x_3^4 \\ 0 & 0 & 2 & 6x_3 & 12x_3^2 & 20x_3^3 \end{pmatrix}$$

then

$$\det M(1,2,3) \;=\; 2(x_2-x_1)^2(x_3-x_1)^3(x_3-x_2)^6$$

The following corollary is exactly what is required for computing $\dim(\mathbf{W}_\mathbf{q}^+)$.

Corollary 10 *Let $N = 2n$, and let $Q(n)$ denote the $N \times N$ matrix whose $2i$th and $(2i+1)$st rows are $\rho_N(x_i, 0)$ and $x_i\rho_N(x_i, 1)$, respectively. Then*

$$\det Q(n) \;=\; \prod_{i=1}^{n} x_i \prod_{1 \le i < j \le n} (x_j - x_i)^4$$

For example, if $n = 3$ we have

$$\begin{aligned} \det Q(3) \;&=\; \det \begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 0 & x_1 & 2x_1^2 & 3x_1^3 & 4x_1^4 & 5x_1^5 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 0 & x_2 & 2x_2^2 & 3x_2^3 & 4x_2^4 & 5x_2^5 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \\ 0 & x_3 & 2x_3^2 & 3x_3^3 & 4x_3^4 & 5x_3^5 \end{pmatrix} \\ &=\; x_1x_2x_3 \det M(2,2,2) \\ &=\; x_1x_2x_3(x_2-x_1)^4(x_3-x_1)^4(x_3-x_2)^4 \end{aligned}$$

Corollary 10 shows that a collection of functions of the form (20) is linearly independent if and only if the corresponding θ_i are distinct, and the stated result for $\dim(\mathbf{W}_\mathbf{q}^+)$ follows. The value of $\dim(\mathbf{W}_{\mathbf{p},\mathbf{q}}^{++})$ is obtained in a similar fashion, using a result analogous to Corollary 10 to compute the determinant of matrices of type $M(1,\ldots,1,2,\ldots,2)$, where there are p occurrences of 1 and q occurrences of 2. We omit the details of this last step, which completes the proof of Theorem 7.

Theorem 7 describes relations among closed form expressions of type (4), but the proof also yields similar results for expressions of type (3).

Corollary 11 *Under the assumptions of Theorem 3, let* $\mathbf{W}^*$ *denote the space spanned by monomial functions of type (3). Then* $\dim \mathbf{W}^* = |S| + 2|T|$*, where* S *is the set of all monomials of the form* $t_1^{a_1} t_2^{a_2} \cdots t_k^{a_k}$ *and* T *is the set of monomials of the form* $t_1^{a_1+b_1} t_2^{a_2+b_2} \cdots t_k^{a_k+b_k}$ *and, for each* i*,* t_i *is one of the roots* $r_1, r_2, \ldots, r_d$*.*

We omit the proof, which is essentially the same as the proof of Theorem 7.

Corollary 12 *Under the assumptions of Theorem 3, the monomial functions of type (3) are linearly independent if and only if formally distinct monomials in* $S \cup T$ *correspond to distinct complex numbers.*

We note that $S \cup T$ is a subset of the set of monomials corresponding to Theorem 7, and thus we obtain the following result.

Corollary 13 *Under the assumptions of Theorem 3, if the monomial functions of type (4) are linearly independent, then so are the monomial functions of type (3).*

We will now apply these results to some of the formulæ in Sections 3.1 and 3.2.

Corollary 14 *For the Fibonacci sum* $f(n)$ *appearing in (9), formula (10) gives the unique* F*-closed formula of type (3) and (11) gives the unique* F*-closed formula of type (4).*

Proof: By Theorem 7, we need only check that, if r_1 and r_2 denote the roots of the Fibonacci recurrence, then

$$r_1^2, r_1 r_2, r_2^2, r_1^3, r_1^2 r_2, r_1 r_2^2, \text{ and } r_2^3$$

are distinct real numbers. This is an elementary calculation.

Corollary 15 *For the sum* $g(n) = \sum_{j=0}^{n-1} G(j)^2$ *arising in the subword avoidance problem with* $A = 2$*, solutions* $g(n)$ *of type (4) are all given by (14) plus constant multiples of relation (15).*

Proof: The roots of the recurrence equation $t^3 - 2t^2 + 1 = 0$ are r_1, r_2, r_3, where r_1 and r_2 are roots of the Fibonacci recurrence and $r_3 = 1$. By Theorem 7, the dimension of the space $\mathbf{W}_{\mathbf{2,0}}^{++}$ spanned by the six degree-2 monomials in $G(n)$ and $G(n+1)$ together with n and 1 is equal to $|S_2| + 2|S_0| - |S_2 \cap S_0|$, where $S_2 = \{r_1^2, r_2^2, 1, r_1 r_2, r_1, r_2\}$, and $S_0 = \{1\}$. Elementary calculation shows that this dimension is equal to 7, hence the monomials generating $\mathbf{W}_{\mathbf{2,0}}^{++}$ are linearly independent apart from a one-parameter family of relations.

Next we consider the case $A = 3$, as a warmup for the general case $A > 2$.

Corollary 16 *For the sum $g(n) = \sum_{j=0}^{n-1} G(j)^2$ arising in the subword avoidance problem with $A = 3$, formula (13) gives the unique G-closed formula of type (3).*

Proof: Here the recurrence equation is $t^3 - 3t^2 + 1 = 0$, which has roots $r_1 = 1 + \eta + \eta^{17}, r_2 = 1 + \eta^7 + \eta^{11}, r_3 = 1 + \eta^5 + \eta^{13}$, where $\eta = e^{2\pi i/18}$ is an 18th root of unity. Again, by Theorem 7, the dimension of the space of monomials $\mathbf{W}^{++}_{\mathbf{2,0}}$ is equal to $|S_2| + 2|S_0| - |S_2 \cap S_0|$, where $S_2 = \{r_1^2, r_2^2, r_3^2, r_1r_2, r_1r_3, r_2r_3\}$, and $S_0 = \{1\}$. A slightly less elementary calculation shows that the formal monomials in S_2, S_0, and $S_2 \cup S_0$ are distinct, so that $\dim(\mathbf{W}^{++}_{\mathbf{2,0}}) = 8$ and the monomial functions generating $\mathbf{W}^{++}_{\mathbf{2,0}}$ are linearly independent.

Corollary 17 *For the more general power sum $g(n) = \sum_{j=0}^{n-1} G(j)^p$ arising in the subword avoidance problem, with $p > 0$ and any $A > 2$, solutions of type (3) are unique if and only if $p \not\equiv 0$ mod 6.*

Proof: An argument analogous to the calculation in Section 3.2 shows that formulæ of type (3) exist expressing $g(n)$ as linear combinations of monomials in $G(n), G(n+1)$, and $G(n+2)$ of degree p, together with 1 and n. We need to compute the dimension of $\mathbf{W}^{++}_{\mathbf{p,0}}$, which by Theorem 7 is equal to $|S_p| + 2|S_0| - |S_2 \cap S_0|$, where $S_0 = \{1\}$ and S_p is the set of all degree-p monomials in r_1, r_2, and r_3, where r_1, r_2, and r_3 are roots of the recurrence equation $t^3 - At^2 + 1 = 0$.

If p is not divisible by 6, the proof will be complete if we can show that formally distinct monomials in S_p evaluate to distinct complex (actually real) numbers, and none of them equals 1. Suppose that $r_1^{e_1} r_2^{e_2} r_3^{e_3} = r_1^{f_1} r_2^{f_2} r_3^{f_3}$, where $\sum e_i = \sum f_i = p$ and $e_i \neq f_i$ for some i. Then by cancellation we obtain the relation $r_i^{u_i} = r_j^{u_j} r_k^{u_k}$ for some rearrangement of the indices, with $u_i, u_j, u_k \geq 0$ and at least one of these exponents positive. Using the relation $r_1r_2r_3 = -1$, if necessary, to eliminate one of the roots, we obtain (after possibly reindexing), $r_i^{v_i} = \pm r_j^{v_j}$ with $v_i, v_j \geq 0$ and at least one of these exponents positive.

It is a straightforward exercise to show that the roots r_1, r_2 and r_3 are all real, and that, if they are arranged in decreasing order, then $r_1 > 1, 0 < r_2 < 1$, and $-1 < r_3 < 0$. From elementary Galois theory we know that there exists an automorphism Φ of the field $K = \mathbf{Q}(r_1, r_2, r_3)$ such that $\Phi : r_1 \mapsto r_2 \mapsto r_3 \mapsto r_1$, i.e., it permutes the roots cyclically. Hence the equation $r_i^{v_i} = \pm r_j^{v_j}$ holds for all three cyclic permutations of the roots. At least one of these equations leads to a contradiction, since $|r_1| > 1$ and $|r_2|, |r_3| < 1$. This proves that formally distinct monomials are distinct, and it remains to show that none can equal 1.

Suppose that $r_1^{e_1} r_2^{e_2} r_3^{e_3} = 1$, and the exponents e_i are not all equal. Applying the identity $r_1r_2r_3 = -1$ we obtain a relation of the form $r_i^{u_i} r_j^{u_j} = \pm 1$ for some pair of distinct i, j, with $u_i, u_j \geq 0$ and at least one positive. Again, this relation holds for all cyclic permutations of the indices, and consideration of absolute values leads to a contradiction in at least one case.

Consequently, we must have $e_1 = e_2 = e_3 = e$ for some e. From the relations $r_1r_2r_3 = -1$ and $(r_1r_2r_3)^e = 1$ we conclude that e is even, which implies that p is a multiple of 6. This completes the proof that monomials in the F are linearly independent when $p \not\equiv 0 \bmod 6$. When $p = 6m$ the relation $(r_1r_2r_3)^{2m} = 1$ gives relations in the F of degree 6, and so the proof of Corollary 17 is complete.

6 Hyperdiscriminants

We have seen in the above theorems and corollaries that we can decide the uniqueness of representations of certain sequences in closed form if we can decide whether or not the $N = \binom{n+d-1}{d}$ formally distinct monomials of degree d in the roots $r_1, \ldots, r_n$ actually are all different, when evaluated as complex numbers. It is worthwhile asking whether there is any general machinery for answering such questions, and, especially, whether the answers can be obtained without explicitly computing the roots r_i.

In principle, such machinery exists, via a small generalization of the ordinary notion of discriminants. Recall that, if f is a polynomial of degree n, we can test whether its roots are distinct by computing a polynomial $\Delta(f)$ in its coefficients, and testing whether $\Delta(f) = 0$. In order to generalize this, the first step is to find polynomial g of degree N whose roots are the N monomials of degree d in the roots of f. Then we need only compute the ordinary discriminant of that polynomial.

To find the polynomial g, we find its coefficients, which are the elementary symmetric functions of the N monomials of degree d in the roots of f. To find those elementary symmetric functions, observe that the elementary symmetric functions of these monomials in the r_i's are symmetric functions in the r_i's themselves. Since any symmetric function of the roots of f can be computed rationally in terms of the coefficients of f, it follows that the coefficients of our polynomial g can be so computed.

The resulting "hyperdiscriminant" $\Delta^d(f) = \Delta(g)$ has the property that it depends only on the original coefficients of f, and $\Delta^d(f) \neq 0$ if and only if all monomials of degree d are distinct.

We have calculated a few of these hyperdiscriminants, and they show some interesting patterns of factorization. For example if $f(x) = x^3 + e_1x^2 + e_2x + e_3$ then the discriminant of it's monomials of degree 2 is

$$-(e_1\,e_2 - e_3)^2\,{e_3}^6\left(-{e_2}^3 + {e_1}^3\,e_3\right)^2\left(4\,{e_2}^3 - {e_1}^2\,{e_2}^2 + 4\,{e_1}^3\,e_3 - 18\,e_1\,e_2\,e_3 + 27\,{e_3}^2\right)^4.$$

In this expression, the third factor is $\Delta(f)$, the ordinary cubic discriminant. The other significant factors can be interpreted as follows:

$$\left(-{e_2}^3 + {e_1}^3 e_3\right) = \prod_{i,j,k} (r_i^2 - r_j r_k), \tag{22}$$

where the product is over all distinct i, j, k, and

$$-\left(e_1 e_2 - e_3\right)^2 = \prod_{i<j} (r_i^2 - r_j^2)/\Delta_3. \tag{23}$$

It is clear that a factorization of $\Delta^d(f)$ with terms analogous to those explained by (22) and (23) will exist for all values of n and d. However, we have not obtained a precise form of these factors in general, and hence our computations do not suffice to explain more than a few sporadic cases of the results given in Sections 4 and 5.

Expressing general products such as (22) and (23) in terms of the elementary symmetric functions e_i would be an interesting question for future research.

7 Concluding remarks

The techniques in this paper can be generalized in a variety of ways. For example, a more careful analysis allows one to drop the assumption that the recurrence polynomial has distinct roots. It is also possible to consider more general summations of the form

$$f(n) = \sum_{j=0}^{n-1} \alpha^j \phi(j) F_1(a_1 n + b_1 j + c_1) F_2(a_2 n + b_2 j + c_2) \cdots F_k(a_k n + b_k j + c_k),$$

where α is a constant, $\phi(j)$ is a polynomial function of j, and $F_1, F_2, \ldots, F_k$ are (possibly distinct) functions, each solving a linear recurrence with constant coefficients. It is straightforward (though somewhat tedious in the most general case) to identify sets of target monomials analogous to (3) and (4), and the process is easily automated. We will not attempt to give general statements of these results, but will simply illustrate them with two more examples.

7.1 A mixed convolution

Let $F(n)$ denote the nth Fibonacci number, and let $G(n)$ be defined by the subword-avoiding recurrence (12) with $A = 3$, in other words $G(0) = 1, G(1) = 3, G(2) = 9$, and $G(n) = 3G(n-1) - G(n-3)$ for $n > 2$. Then we have the following identity:

$$\sum_{j=0}^{n} j\, F(j) G(n-j) \;=\; 18G(n+1) - (9G(n) + 5G(n+2) + 3F(n) + nF(n+1))$$

The target monomials in this case are

$$F(n), nF(n), F(n+1), nF(n+1), G(n), n^2G(n), G(n+1), n^2G(n+1), G(n+2), n^2G(n+2)$$

and the (unique) solution is obtained by solving a system of 10 equations in 10 unknowns.

7.2 A partial sum

Consider the sum

$$\sum_{j=0}^{n-1} F(j)x^j$$

where $F(n)$ is a Fibonacci number and x is an indeterminate. The summand is a product of two C-finite sequences, one of degree two and the other of degree one. A quick calculation with target monomials $1, F(n)x^n$, and $F(n+1)x^n$ produces the identity

$$\sum_{j=0}^{n-1} F(j)x^j = \frac{x}{1-x-x^2} - x^n\left(\frac{1-x}{1-x-x^2}F(n) + \frac{x}{1-x-x^2}F(n+1)\right),$$

which quantifies the remainder term in the Fibonacci generating function (this result appears as problem 1.2.8.21 in [3]). Our approach can be easily extended, for example, using $1, F(n)^2x^n, F(n)F(n+1)x^n$, and $F(n+1)^2x^n$ as target monomials and solving four equations in four unknowns, we obtain the partial summation formula

$$\sum_{j=0}^{n-1} F(j)^2x^j = \frac{x(1-x)}{1-2x-2x^2+x^3} - x^n\,R_n(x) \tag{24}$$

where

$$R_n(x) = \frac{(1-2x-x^2)F(n)^2 + 2x^2F(n)F(n+1) + x(1-x)F(n+1)^2}{1-2x-2x^2+x^3} \tag{25}$$

The first term in (24) is the full generating function for squares of Fibonacci numbers, and a formula for the full generating function for all powers p appears in [6] (see also [3], problem 1.2.8.30).

We note in conclusion, that to obtain (24) and (25) by this method it was only necessary to know the first four values of the sum, and also that F satisfies some 2-term recurrence with constant coefficients.

References

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