Some unsolved problems

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Here are some mathematical problems that are, as far as I know, unsolved, and which I have encountered in recent work.

1 Series for π

A great many rapidly converging series for π are known. Most often they are of the form

$$\pi = \sum_{n \geq 0} t_n,$$

where t_n is a hypergeometric term, that is, t_{n+1}/t_n is a rational function of n. Our question roughly is this - how fast can such a series converge to π?

Of course without further conditions the question is trivial, so we must add that the hypergeometric term t_n has rational coefficients. The known series all seem to converge exponentially fast, that is, $C = \lim t_n^{1/n}$ exists and is finite and nonzero. Given any such series it is simple to construct another one in which C is replaced by C^2, so there exist such representations of π in which the constant C is arbitrarily small. But can it be 0? That is, can such a series converge superexponentially fast?

We ask the question precisely as follows.

Does there exist an entire function $f(z) = \sum_{n \geq 0} a_n z^n$ such that the coefficients a_n are hypergeometric terms over the rational numbers, and $f(1) = \pi$?

It is worth noting that if we replace "π" by "e" in the above it becomes quite trivial, since e^z is an entire function.

2 Growth of partition functions

Let S be a set of positive integers and let M be a set of nonnegative integers containing 0. Let $p(S, M; n)$ denote the number of partitions of n whose parts all lie in S and the multiplicities of whose parts all lie in M. The following question was encountered in some joint work with Rod Canfield.

Can we choose S and M so that

1. *S and M are both infinite sets, and*
2. *all sufficiently large integers n are represented, i.e., $p(S, M; n) > 0$ for all large n, and*

3. $p(S,M;n)$ is of at most polynomial growth in n?

3 A problem in asymptotics

Let

$$f(z)=\frac{\arctan\sqrt{2e^{-z}-1}}{\sqrt{2e^{-z}-1}}.$$

If $f(z)=\sum_{n\geq 0}a_n z^n$, find the first term of the asymptotic behavior of the a_n's.

Recent progress: Mark Ward has found [1] a complete expansion of these coefficients. It's not quite an asymptotic series in the usual sense, but it is probably the best that can be done, given the oscillatory nature of the terms.

4 The quadratic character of binomial coefficients

An integer n is p-balanced if, among the nonzero values of $\{\binom{n}{k}\}_{k=0}^{n}$ (mod p) there are equal numbers of quadratic residues and quadratic nonresidues mod p. Say that the prime p is special if no integer n, $0\leq n<p$ is p-balanced. The primes 2, 3, 11 are special, and no other primes $<1,000,000$ are special.Are there any more special primes? See my paper with Richard Garfield, *The distribution of the binomial coefficients modulo p*, J. Number Theory **41** (1992), 1-5, which is available from my web site.

5 Young tableaux

In 1992 I found[1] a relationship between the numbers of standard Young tableaux of n cells whose first row has length $\leq k$, on the one hand, and the number of permutations of n letters whose longest ascending subsequence has length $\leq k$, on the other.

More precisely, let k be an even number, let $y_k(n)$ be the number of Young tableaux of n cells whose first row has length $\leq k$, and let $u_k(n)$ be the number of permutations of n letters that have no ascending subsequence of length $>k$. Then it is true that

$$\binom{2n}{n}u_k(n)=\sum_r\binom{2n}{r}(-1)^r y_k(r)y_k(2n-r).$$

I discovered this relationship by using analytical methods, but the relationship itself betrays none of its analytical origins, and is in fact a purely combinatorial, finite relation. That being the case, it should have a purely combinatorial derivation. Find one.

[1] Ascending subsequences of permutations and the shapes of Young tableaux, J. Combinatorial Theory, Ser. A **60** (1992), 155-157.

6 Distinct multiplicities

Let $T(n)$ be the set of partitions of n for which the (nonzero) multiplicities of its parts are all different, and write $f(n) = |T(n)|$. See Sloane's sequence `A098859` for a table of values. Find any interesting theorems about $f(n)$. The mapping that sends a partition of n to another partition of n in which the roles of parts and multiplicities are interchanged is a well defined involution on $T(n)$, which is how I arrived at the study of this problem.

7 Toeplitz determinants

Find $f(n)$, the number of monomials in the expansion of the $n \times n$ general Toeplitz determinant

$$\det \left(\left(a_{|i-j|} \right)_{i,j=1,\ldots,n} \right).$$

See Sloane's sequence `A019447` for the values up to $n = 11$. Is $f(n)$ of superexponential growth?

8 Chromatic number

By a theorem of the late George Szekeres and myself[2], we have

$$\chi(G) \leq 1 + \max_{G' \subseteq G} \delta(G'),$$

where χ is the chromatic number and δ is the minimum degree of the vertices. For precisely which graphs G does the sign of equality hold?

Recent progress: In April, 2010, Xuding Zhu proved the following: Given a graph G. To determine if $\chi(G) < 1 + \max_{G' \subseteq G} \delta(G')$ is NP-complete. Possibly there might be a verbal description of the case of equality anyhow, which might be of some theoretical use.

References

[1] Mark Daniel Ward, Asymptotic Rational Approximation To Pi: Solution of an "Unsolved Problem" Posed By Herbert Wilf, Discrete Mathematics and Theoretical Computer Science, Vol. AM (2010), 591–602.

[2] An inequality for the chromatic number of a graph, (with G. Szekeres), *J.* Combinatorial Theory, **4** (1968), 1-3.