Due: Thursday September 26 at the end of class.

(1) Show that the following 3 definitions for the smoothness of a vector field $Y$ are equivalent:
   (a) Locally in a coordinate $(U,x)$, $Y$ has the form $Y(p) = \sum a_i(p)\frac{\partial}{\partial x_i}|_p$ where $a_i: U \to \mathbb{R}$ is smooth,
   (b) $Y: M \to TM$ is smooth, with $Y(p) \in T_pM$,
   (c) $Y: C^\infty(M) \to C^\infty(M)$ is a derivation, i.e. $Y(fg) = fY(g) + gY(f)$, and the vector field is defined by $Y(p)(f) = Y(f)(p)$.

(2) Show that $[X,Y](f) = X(Y(f)) - Y(X(f))$ defines a smooth vector field if $X,Y$ are smooth vector fields, and that it has the following properties:
   (a) $[fX,Y] = f[X,Y] - Y(f)X$ and $[X,fY] = f[X,Y] + X(f)Y$.
   (b) $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$ (the Jacobi identity).
   (c) Show hat if $Y(q) = 0$ if and only if $\phi_t(q) = q$ for all $t$.

(3) Compute the flow of the following vector fields:
   (a) $Z(a,b) = (4a - 3b)\frac{\partial}{\partial a} + (6a - 5b)\frac{\partial}{\partial b}$ on $\mathbb{R}^2$.
   (b) $Z(a) = (1 + a^2)\frac{\partial}{\partial a}$ on $\mathbb{R}$.
   Are the vector fields complete?

(4) Let $f: M^n \to N^n$ be a diffeomorphism and $X,Y$ are vector fields on $M$ resp. $N$.
   (a) Derive a formula for $f_*(X)$ and $f^*(Y)$ as derivations. Why does this show that these are differentiable vector fields?
   (b) Derive a formula for the flow of $f_*(X)$ and $f^*(Y)$ in terms of the flow of $X$ and $Y$.

(5) (Extra Credit)
   Let $f: SL(2,\mathbb{R}) \to \mathbb{R}$ where $f(A) = tr(A)$. Find the regular values of $f$. What is the diffeomorphism type of the surface $f^{-1}(q)$ for $q$ regular? What does $f^{-1}(q)$ look like for $q$ singular?