1. Show that our two definitions of orientability of a manifold \( M^n \) are equivalent:
   (a) There exists an atlas \((U_\alpha, x_\alpha)\) such that \(D(x_\alpha \circ x^{-1}_\beta)\) is orientation preserving (i.e. has positive determinant).
   (b) There exists a choice of orientations on \( T_p M \) for all \( p \), i.e. a choice of (equivalence classes) of basis, which represents one of the possible orientations on this vector space, such that it varies smoothly (or continuously) with \( p \). I.e. for each \( p \in M \), there exists a neighborhood \( U \) of \( p \) and smooth (continuous is sufficient) vector fields \( Y_1, \ldots, Y_n \) on \( U \) such that for all \( p \in U \), the basis \( Y_1(p), \ldots, Y_n(p) \) represents the orientation class you chose on \( T_p M \).

2. Let \( \Gamma \) be a group that acts properly discontinuously on a manifold \( M^n \) with quotient \( M/\Gamma \).
   Recall we already showed that \( M/\Gamma \) is a manifold.
   Using the definition of orientability in (1) (b):
   (a) Show that if \( M \) is orientable and \( \Gamma \) acts orientation preserving on \( M \), then \( M/\Gamma \) is orientable as well.
   (b) If \( M/\Gamma \) is orientable, show that there exists an orientation on \( M \) such that \( \Gamma \) acts orientation preserving on \( M \).
   (c) Make precise that \( \mathbb{R}P^n \) is orientable iff \( n \) is odd. I.e. carefully define the orientation of the sphere, and show the restriction of the antipodal map to the sphere is orientation preserving iff \( n \) is odd. Then also make precise again that all lens space are orientable.

3. Let \( F \) and \( G \) be two vector bundles over \( M \) and define the direct sum \( E = F \oplus G \), where the fiber \( E_p \) over \( p \in M \) is the direct sum of the vector spaces \( F_p \) and \( G_p \).
   (a) Show that \( E \) is in fact a vector bundle.
   (b) Define the orientation of a vector bundle, and show that if both \( F \) and \( G \) are orientable, then \( E \) is orientable as well.
   (c) Show that if both \( E \) and \( F \) are orientable, then \( G \) is orientable as well.

4. Show that a one dimensional vector bundle (i.e., the fiber over \( p \in M^n \) is one dimensional) is trivial iff it is orientable.

5. Let \( f: M \to B \) be a submersion.
   (a) Show that \( TM = f^*(TB) \oplus V \) where \( f^*(TB) \) is the ”pull back” vector bundle over \( M \) whose fiber over \( p \in M \) is \( T_{f(p)}B \), and \( V \) is the ”vertical” bundle over \( M \) whose fiber over \( p \in M \) is the tangent space of the fiber through \( p \), i.e. \( T_p(f^{-1}(f(p))) \).
   (b) Show that \( V \) is isomorphic to the vector bundle \( \ker D(f) \) and \( f^*(TB) \) to the normal bundle, if we endow \( M \) with a Riemannian metric.
   (c) Show that the fibers of \( f \) are orientable if \( M \) and \( B \) are. In particular, the Brieskorn varieties from the last assignment are orientable.

6. (Extra credit 1)
   Find a base \( B \), and two vector bundles \( F \) and \( G \) over \( B \), such that both \( F \) and \( G \) are non-orientable, but \( E = F \oplus G \) is non-orientable as well.
(7) (Extra credit 2)

For those who know what the fundamental group $\pi_1(M)$ is. If you do not, you can look up the definition, which is not difficult.
(a) Construct a map $O : \pi_1(M) \to \mathbb{Z}_2$.
(b) Make this definition rigorous and show it is well defined and a homomorphism.
(c) Show that $O$ is trivial iff $M$ is orientable.

Comment: Next semester you will learn that a homomorphism $\alpha : \pi_1(M) \to \mathbb{Z}_2$ is an element of $H^1(M, \mathbb{Z}_2)$, which is called the first Stiefel Whitney class.