Due: Thursday November 14 at the end of class.

(1) Let $\Lambda(V^*) = \bigoplus \Lambda_k(V^*)$ be the exterior algebra with its natural wedge product, and $A_k(V)$ the vector space of alternating multilinear forms of degree $k$. Recall that we identified $\Lambda_k(V^*)$ with $A_k(V)$ by sending $f_1 \wedge f_2 \wedge \cdots \wedge f_k \in \Lambda(V^*)$ to the multilinear form $A(v_1, v_2, \cdots, v_k) = \det(f_i(v_j))$.

(a) Show that this is a well defined linear map and an isomorphism.
(b) Show that under this isomorphism the god given wedge product on $\Lambda(V^*)$ becomes the wedge product on $A_k(V)$ defined by

$$(F \wedge G)(v_1, \ldots, v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} \text{sign}(\sigma) F(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) G(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

for $F \in A_k(V)$ and $G \in A_{\ell}(V)$.
(c) Show that this wedge product on $A(V)$ is the same as the one defined in Lee’s book, i.e.,

$$F \wedge G = \frac{(k + \ell)!}{k!\ell!} \text{Alt}(F \otimes G)$$

Notice that in our language Lemma 9.6 in Lee’s book becomes a triviality.

(2) Let $V$ be an n-dimensional vector space. We say that $w \in \Lambda_k(V)$ is decomposable if $w = v_1 \wedge \cdots \wedge v_k$ for some $v_i \in V$.

(a) Show that every element in $\Lambda_1(V)$ and $\Lambda_{n-1}(V)$ is decomposable.
(b) If $\dim V = 4$, show that $\omega \in \Lambda_2(V)$ is decomposable iff $\omega \wedge \omega = 0$.
(c) If $\dim V = 4$, find an $\omega \in \Lambda_2(V)$ which is not decomposable.

Hint: Show that given $\omega \in \Lambda_2(V)$ with $\dim V = 4$, there there exists an orthonormal basis $e_1, e_2, e_3, e_4$ such that $\omega = ae_1 \wedge e_2 + be_3 \wedge e_4$.

(3) Let $G_2(V)$ be the Grassmannian of 2-planes in $V$ (a compact manifold).

(a) If $v_i \in V$, $i = 1, \ldots, k$ show that $v_1 \wedge v_2 \wedge \cdots \wedge v_k \neq 0$ iff $v_1, \cdots, v_k$ are linearly independent.
(b) Let $D$ be the set of decomposable elements in $\Lambda_2(V)$ moduli scaling, i.e.,

$$D = \{ \omega = v_1 \wedge v_2 \mid v_1, v_2 \in V \}/(\omega \sim \lambda \omega).$$

Show that the map $D \to G_2(V)$ sending $v_1 \wedge v_2 \in D$ to the 2-plane spanned by $v_1$ and $v_2$ is a bijection.

(Extra Credit) Show that

a) $S^1 \times SU(n)$ and $U(n)$ have the same Lie algebra.
b) Show that $U(n)$ is diffeomorphic to $S^1 \times SU(n)$ but not isomorphic as Lie groups.