Each problem is worth 10 points (partial credit will be given). You can use Lee’s book, your class notes and your homework as a reference. You are not allowed to use the internet, or work with other students. Be as rigorous as possible! Write legibly and well organized! The exam should be turned in to me at the beginning of class on Thursday.

(1) Let $\Gamma \subset O(2n + 1)$ be a group that acts freely on $S^{2n}$. Show that $\Gamma = \{ \pm \text{Id} \}$.

(2) Show that there is no smooth immersion $f: S^n \to \mathbb{R}^n$.

(3) Consider the following vector fields on $\mathbb{R}^3$:

$$X = z \frac{\partial}{\partial x} - y \frac{\partial}{\partial z}, \quad Y = -z \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}$$

Explicitly compute the following:

(a) The flow $\phi_t$ of $X$.
(b) The Lie bracket $[X, Y]$.

(4) For the vector fields $X, Y$ in the previous problem, compute, for a fixed value of $t$, the vector field $\phi_t^* (Y)$ and verify that $[X, Y] = \frac{d}{dt}|_{t=0}(\phi_t^* (Y))$.

(5) Given two manifolds $M$ and $N$, when is the product $M \times N$ orientable? Apply this to $S^2 \times S^2$, $\mathbb{RP}^2 \times S^2$ and $\mathbb{RP}^2 \times \mathbb{RP}^2$.

(6) (Extra Credit)
Let $f: SL(2, \mathbb{R}) \to \mathbb{R}$ where $f(A) = tr(A)$. Find the regular values of $f$. Determine the diffeomorphism type of the surfaces $f^{-1}(q)$ for $q$ regular.