12.6

Quadric surfaces

Review:

Vector equation of a line L: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$ $\mathbf{r}_0$ point on the line, direction $\mathbf{v} = \langle a, b, c \rangle$

Parametric (scalar) equation of a line L: $x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct$

$\mathbf{r}(t) = P_0 + t(P_1 - P_0)$ with $0 \le t \le 1$ is the line which goes from P_0 to P_1

Vector equation of a plane: $\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$

$\mathbf{r}_0$ on the plane

$\mathbf{n} = \langle a, b, c \rangle$ normal to plane

Scalar equation of the plane: $ax + by + cz = d$

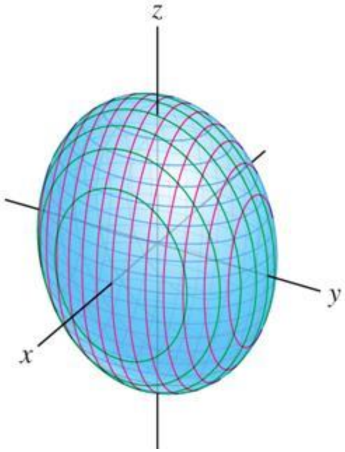
Angle between two planes with normal $\mathbf{n}_1, \mathbf{n}_2$: $\cos\theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$

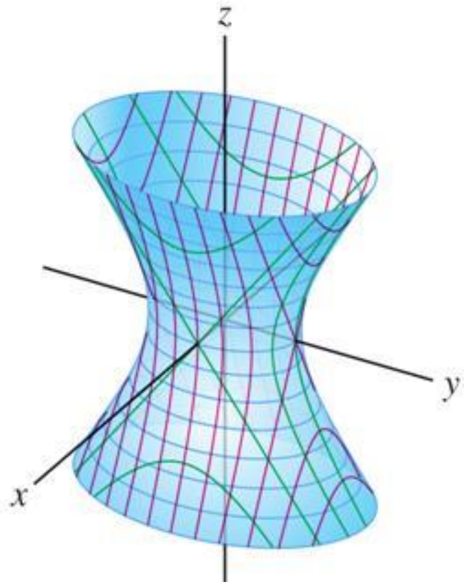
Distance between a point Q and a plane: $\dfrac{|(Q - \mathbf{r}_0) \cdot \mathbf{n}|}{|\mathbf{n}|}$

Distance between a point Q and a line: $\dfrac{|(Q - \mathbf{r}_0) \times \mathbf{v}|}{|\mathbf{v}|}$

Surfaces in 3-space:

Ellipsoid $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



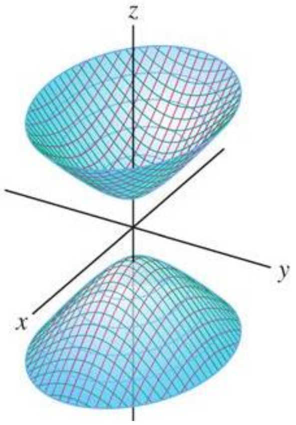


Hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$z=k:\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$ any k

horizontal traces are ellipses



Hyperboloid of two sheet

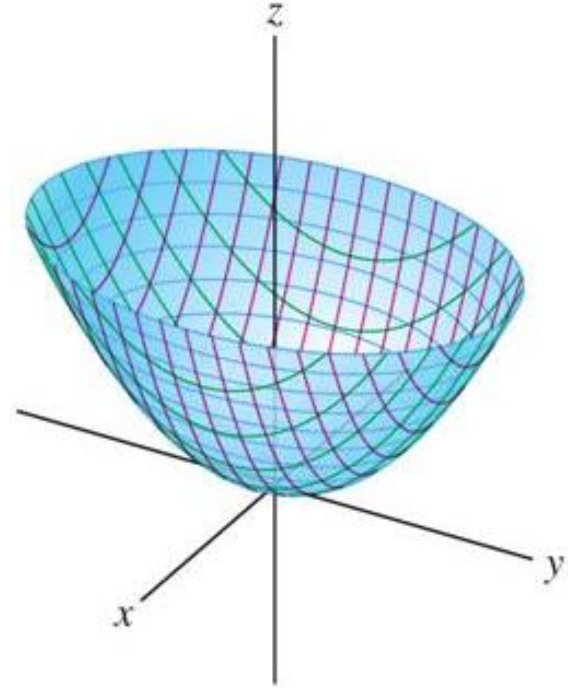
$$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$z=k:\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}-1,\ \ |k|\ >1$

horizontal traces are ellipses

(Elliptical paraboloid) $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

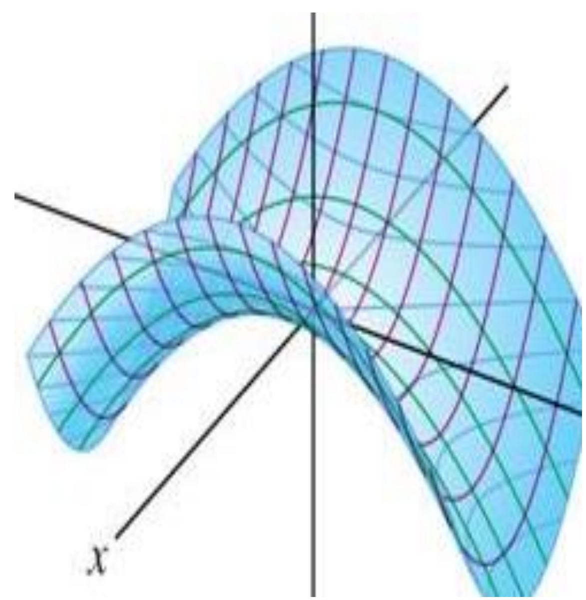
Horizontal traces are ellipses (notice $z > 0$)



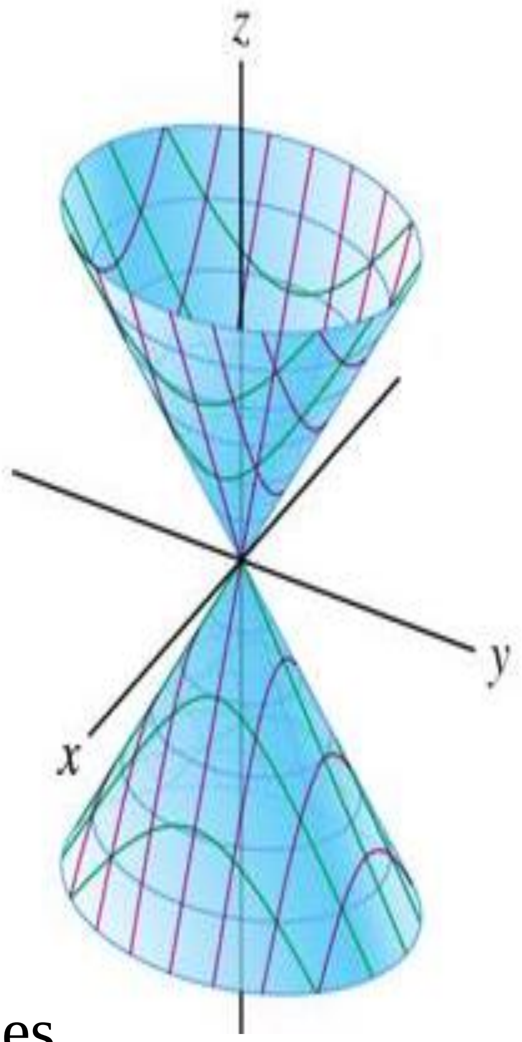
(Hyperbolic paraboloid) $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

Horizontal traces are hyperbolas

Vertical traces are parabolas



Cone $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



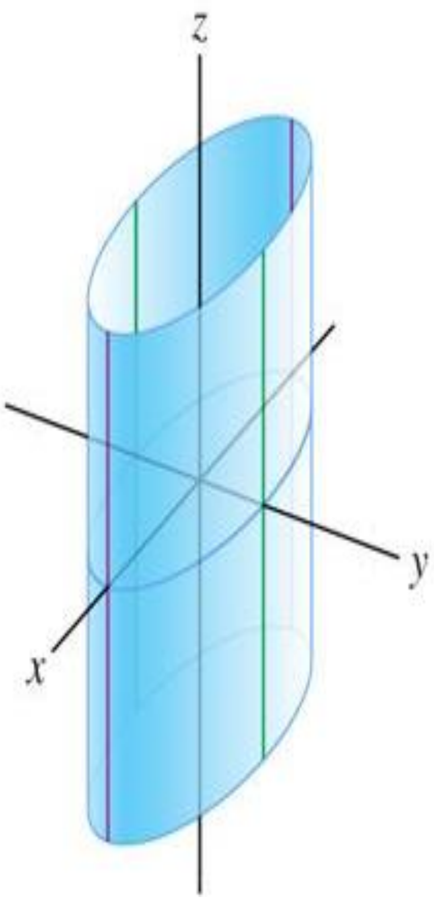
Horizontal traces are ellipses

Vertical traces are hyperbolas

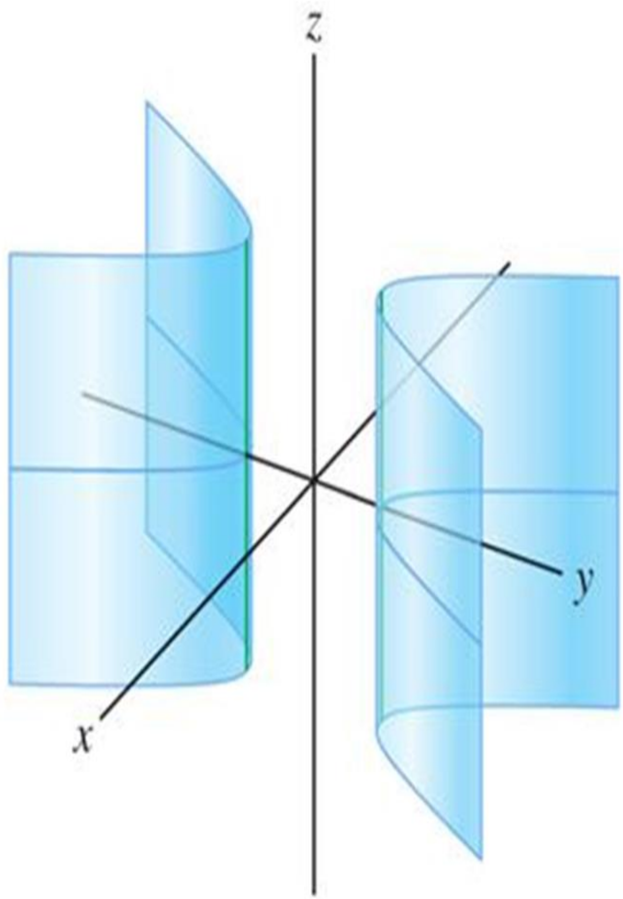
Notice: If $x = 0$ $z = \pm \frac{y}{b}$ straight lines

Cylinders:

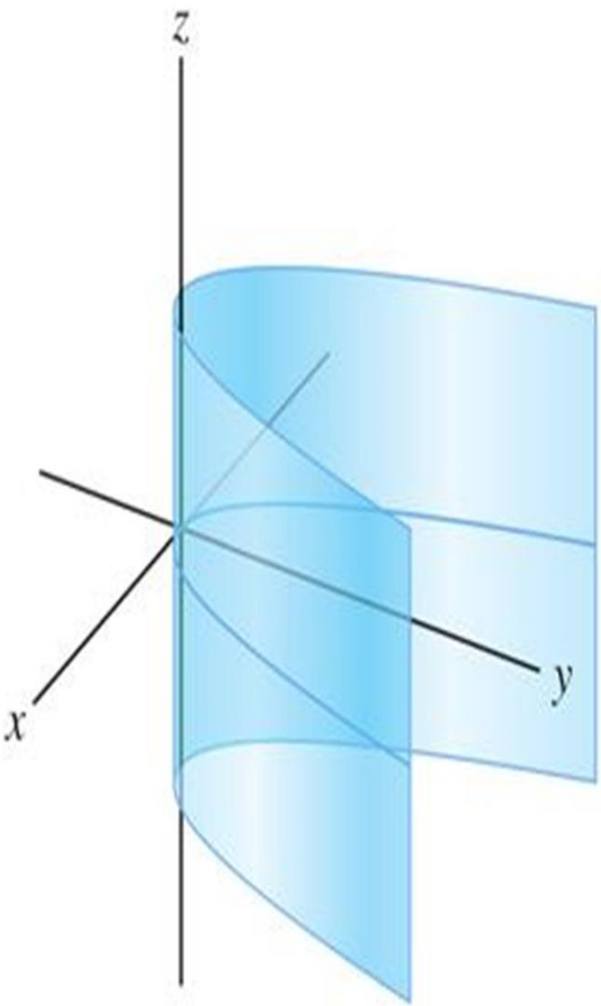
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$



$$y=ax^2$$



21–28 Match the equation with its graph (labeled I–VIII). Give reasons for your choices.

21. $x^2 + 4y^2 + 9z^2 = 1$ **VII**

22. $9x^2 + 4y^2 + z^2 = 1$ **IV**

23. $x^2 - y^2 + z^2 = 1$ **II**

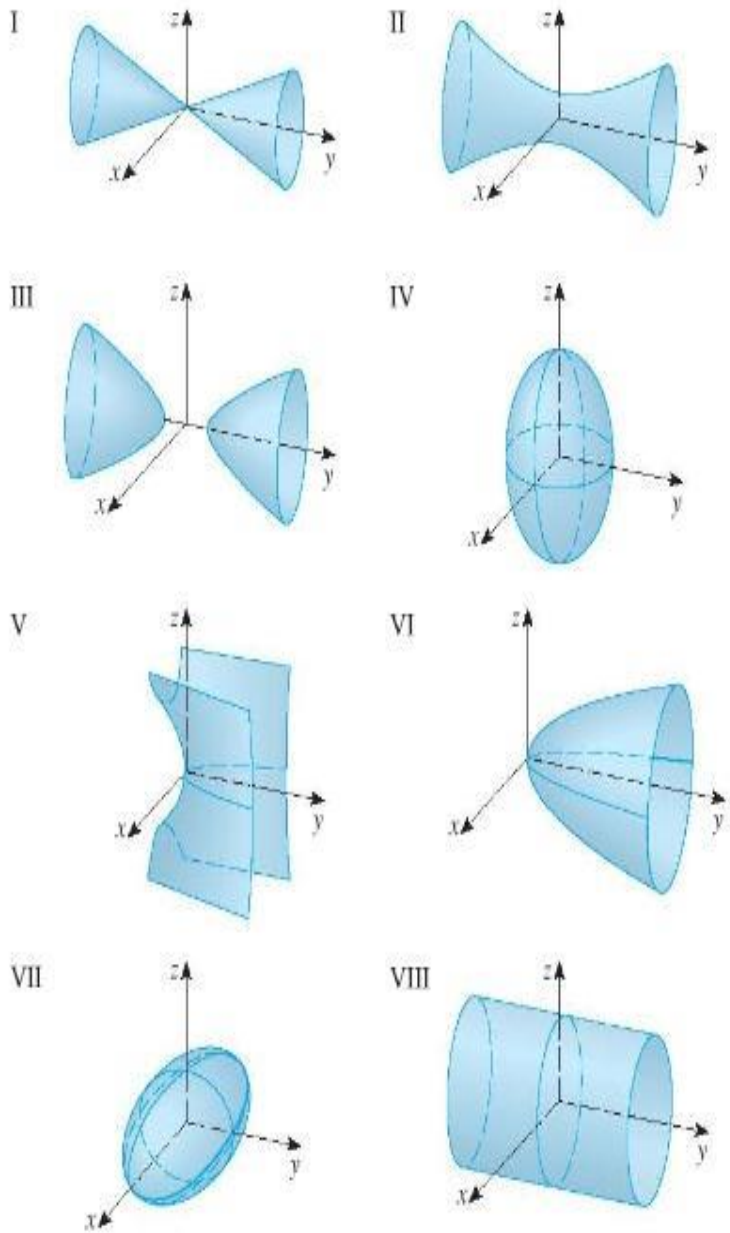
24. $-x^2 + y^2 - z^2 = 1$ **III**

25. $y = 2x^2 + z^2$ **VI**

26. $y^2 = x^2 + 2z^2$ **I**

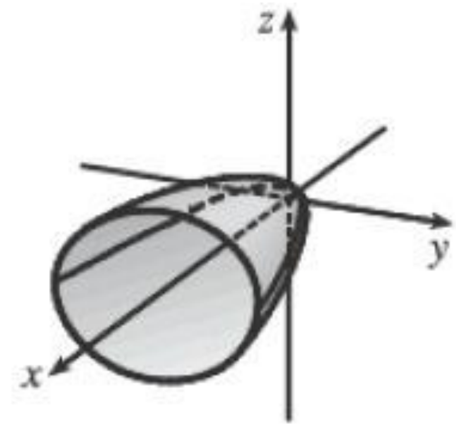
27. $x^2 + 2z^2 = 1$ **VIII**

28. $y = x^2 - z^2$ **V**



Draw a picture of the following surfaces:

$a)\quad x = 2y^2 + 3z^2$ or $\frac{x}{6} = \frac{y^2}{3} + \frac{z^2}{2}$

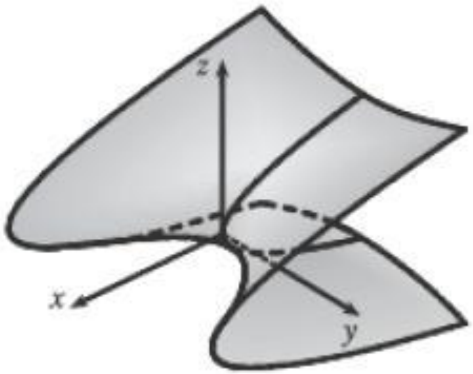


elliptic paraboloid

opens in the positive $x-$direction

$b)\quad 4x - y^2 + 4z^2 = 0$ or $x = \frac{y^2}{4} - \frac{z^2}{1}$

hyperbolic paraboloid



Draw a picture of the surface: $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$

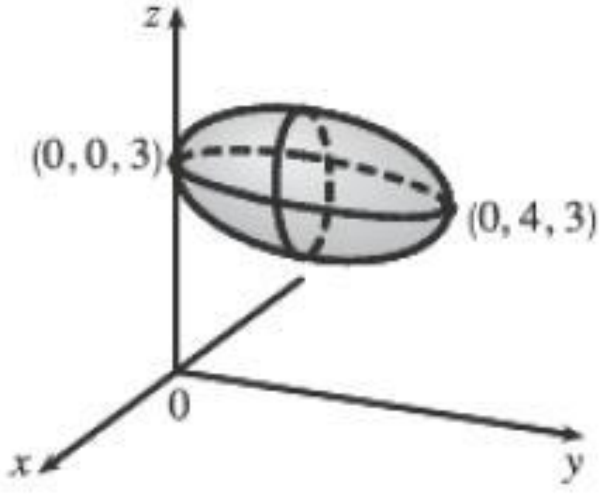
$$4x^2 + \left(y^2 - 4y + \underline{\;\mathbf{4}\;}\right) + 4\left(z^2 - 6z + \underline{\;\mathbf{9}\;}\right) = -36 + \underline{\;\mathbf{4}\;} + \underline{\;\mathbf{36}\;}$$

$$4x^2 + (y-2)^2 + 4(z-3)^2 = 4$$

$$\frac{x^2}{1} + \frac{(y-2)^2}{4} + \frac{(z-3)^2}{1} = 1$$

ellipsoid center: $(0, 2, 3)$

notice e.g. $y > 0$



Summary:

Equation	Surface	
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Ellipsoid	all variables present all variables squared
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$	Hyperboloid of one sheet	
$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$	Hyperboloid of two sheets	
$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	Cone $\left(\text{elliptic}\right)$	
$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$	Paraboloid $\left(\text{elliptic}\right)$	all variables present one variable not squared
$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$	Paraboloid $\left(\text{hyperbolic}\right)$	

one variable not present $\Rightarrow$ cylinder opening in the direction of the missing variable

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ Elliptic cylinder $\quad$ $\frac{x^2}{a^2}-\frac{z^2}{b^2}=1$ Hyperbolic cylinder $\quad$ $z=ax^2$ Parabolic cylinder

13.1

Curves in Space

A curve in space (or the plane):

$$x = f(t),\ y = g(t),\ z = h(t) \text{ or}$$

$$\mathbf{r}(t) = \langle f, g, h \rangle = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$$

in the plane:

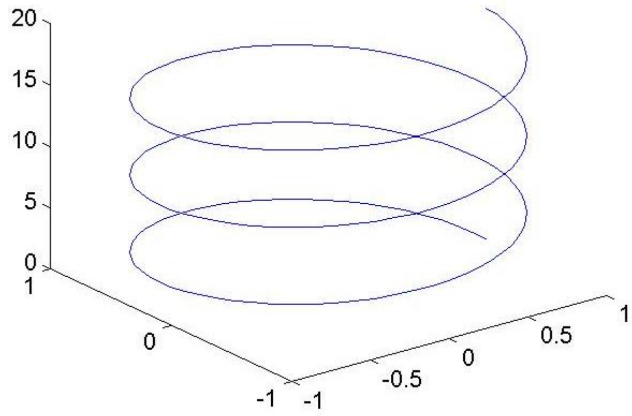
$$\mathbf{r}(t) = f(t)\mathbf{j} + g(t)\mathbf{j}$$

This is a **vector valued** function (as opposed to scalar valued)

Example: $\mathbf{r}(t) = \langle \cos(t), \sin(\text{t}), \text{t} \rangle$

Notice that $x(t)^2 + y(t)^2 = 1$ i.e. the curve lies on a cylinder

A circular helix:



A curve in the plane: $\mathbf{r}(t) = 4\cos(2t)\mathbf{i} + 9\sin(2\,\mathrm{t})\,\mathbf{j}$

Notice that $\frac{x^2}{4^2} + \frac{y^2}{9^2} = 1$ The particle travels along an ellipse

$\mathbf{r}(0) = 4\mathbf{i} = \mathbf{(4,0)}$ Travels counter clockwise

as $0 \le t \le 2\pi$ the particle goes twice around the ellipse

Trefoil knot:

$$\mathbf{r}(t) = \left\langle \left(2 + \cos\left(\tfrac{3t}{2}\right)\right)\cos t, \left(2 + \cos\left(\tfrac{3t}{2}\right)\right)\sin(t), \sin\left(\tfrac{3t}{2}\right) \right\rangle$$

For an animation go to :
http://math.bu.edu/people/paul/225/trefoil_knot.html

For a vector valued function:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Take limits, derivatives and integrals component wise:

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

$$\int_a^b \mathbf{r}(t)\, dt = \left\langle \int_a^b f(t)\, dt, \int_a^b g(t)\, dt, \int_a^b h(t)\, dt \right\rangle$$

Example: Let $\mathbf{r}(t) = \left\langle \frac{\sin(t)}{t}, e^{2t}, \ln(1-t) \right\rangle$. Find $\lim_{t \to 0} \mathbf{r}(t)$

$$\lim_{t \to 0} \mathbf{r}(t) = \left\langle \lim_{t \to 0} \frac{\sin(t)}{t}, e^0, \ln(1) \right\rangle = \langle 1, 1, 0 \rangle = \mathbf{i} + \mathbf{j}$$

Examples:

(a) Let $\mathbf{r}(t)=\left\langle \sin^{-1}t, \sqrt{1-t^2}, \ln(1+3t)\right\rangle$. Find $\mathbf{r}'(t)$

$$\mathbf{r}'(t)=\left\langle \frac{1}{\sqrt{1-t^2}},\ \frac{1}{2}\left(1-t^2\right)^{-1/2}(-2t)\ ,\frac{1}{1+3t}(3)\right\rangle \quad =\left\langle \frac{1}{\sqrt{1-t^2}},\frac{-t}{\sqrt{1-t^2}},\frac{3}{1+3t}\right\rangle$$

(b) $\displaystyle\int_1^2\left(t^2\mathbf{i}+t\sqrt{t-1}\,\mathbf{j}+t\sin(\pi t)\mathbf{k}\right)dt \quad =\left\langle \int_1^2 t^2dt,\int_1^2 t\sqrt{t-1}dt,\int_1^2 t\sin(\pi t)dt\right\rangle$

$$\int_1^2 t^2dt=\left[\frac{t^3}{3}\right]_1^2 =\frac{8}{3}-\frac{1}{3} =\boxed{\frac{7}{3}}$$

$$\int_1^2 t\sqrt{t-1}dt \quad =\int_0^1 (u+1)\sqrt{u}du \quad =\int_0^1\left(u^{3/2}+u^{1/2}\right)dt \quad =\left[\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right]_0^1$$

$u=t-1 \qquad du=dt$

$$=\frac{2}{5}+\frac{2}{3} \quad =\frac{6}{15}+\frac{10}{15} \quad =\boxed{\frac{16}{15}}$$

$t=1\Rightarrow u=0$

$t=2\Rightarrow u=1$

$$\int_1^2 t\sin(\pi t)\,dt \qquad u=t \qquad dv=\sin(\pi t)$$

$$du=dt \qquad v=\frac{-1}{\pi}\cos(\pi t) \qquad \int u\,dv=uv-\int v\,du$$

$$=\left[\frac{-t}{\pi}\cos(\pi t)\right]_1^2-\int_1^2\frac{-1}{\pi}\cos(\pi t)\,dt$$

$$=\left[\frac{-t}{\pi}\cos(\pi t)+\frac{1}{\pi^2}\sin(\pi t)\right]_1^2=\left(\frac{-2}{\pi}\cos(2\pi)+\frac{1}{\pi^2}\sin(2\pi)\right)-\left(\frac{-1}{\pi}\cos(\pi)+\frac{1}{\pi^2}\sin(\pi)\right)$$

$$=\left(\frac{-2}{\pi}(1)+0\right)-\left(\frac{-1}{\pi}(-1)+0\right)=\boxed{\frac{-3}{\pi}}$$

$$\int_1^2\left(t^2\mathbf{i}+t\sqrt{t-1}\,\mathbf{j}+t\sin(\pi t)\mathbf{k}\right)dt=\boxed{\left\langle\frac{7}{3},\frac{16}{15},\frac{-3}{\pi}\right\rangle}$$

Differentiation Rules:

Integration rules:

1. $\frac{d}{dt}\left[\mathbf{u}(t)+\mathbf{v}(t)\right]=\mathbf{u}'(t)+\mathbf{v}'(t)$

$$\int(\mathbf{u}(t)+\mathbf{v}(\mathrm{t}))dt=\int\mathbf{u}(t)dt+\int\mathbf{v}(t)dt$$

2. $\frac{d}{dt}\left[c\mathbf{u}(t)\right]=c\mathbf{u}'(t)$

$$\int c\mathbf{u}(t)dt=c\int\mathbf{u}(t)dt=$$

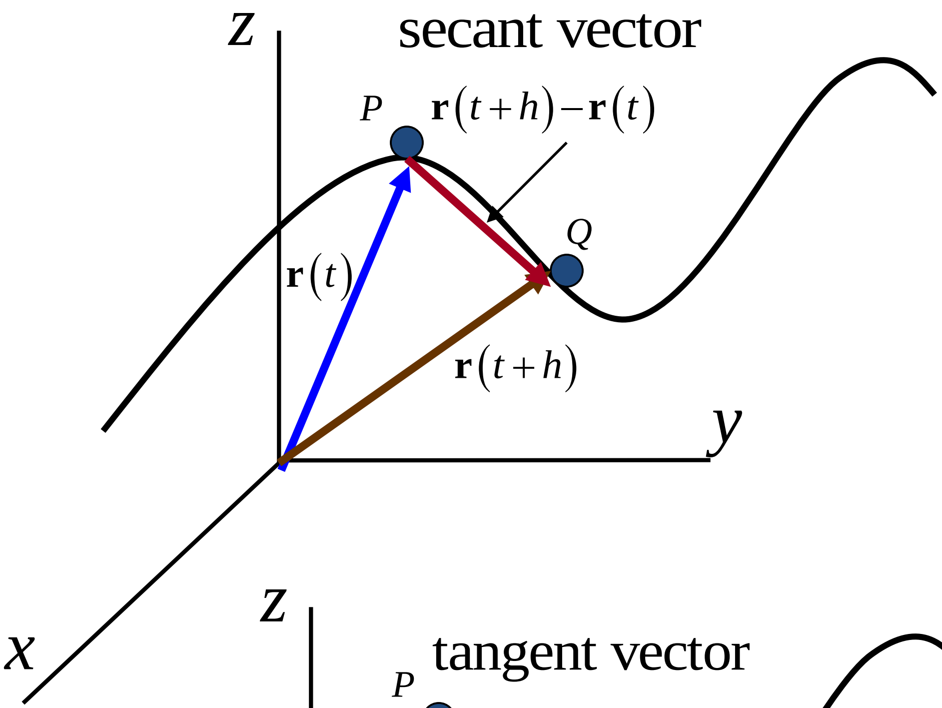
3. $\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right]=f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$

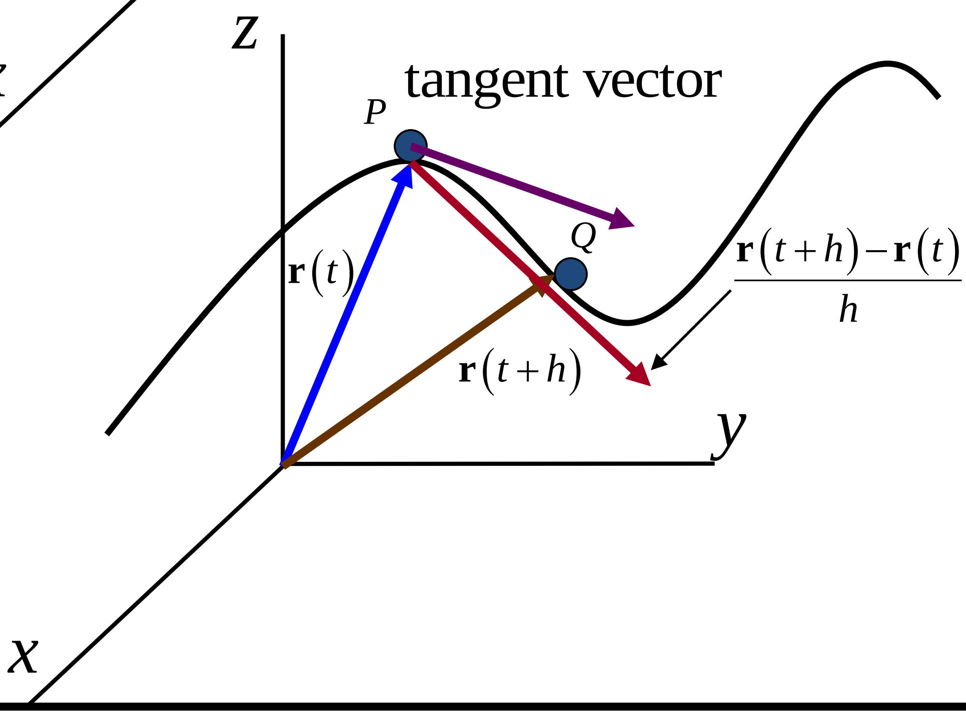
4. $\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]=\mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$

5. $\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$

6. $\frac{d}{dt}\left[\mathbf{u}(f(t))\right]=f'(t)\mathbf{u}'(f(t))$

Geometry of the derivative vector :





Tangent vector

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

The **tangent line** to a smooth curve at $t = t_0$

$$\mathbf{r}(t) = \left\langle f(t), g(t), h(t) \right\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

It passes through the point $\mathbf{r}(t_0) = \left(f(t_0), g(t_0), h(t_0) \right)$,

and has the same direction as the tangent vector at t_0

$$\mathbf{r}'(t_0) = \left\langle f'(t_0), g'(t_0), h'(t_0) \right\rangle.$$

So the equation of the tangent line is:

$$\boxed{\mathbf{s} \to \mathbf{r}(t_0) + s\mathbf{r}'(t_0)}$$

Or in parametric form:

$$x = f(t_0) + sf'(t_0), \quad y = g(t_0) + sg'(t_0), \quad z = h(t_0) + sh'(t_0)$$

Problem : A drunken bee travels along the path $\mathbf{r}(t) = \langle \cos(2t), \sin(2\,\mathrm{t}), \mathrm{t}\rangle$ for 10 seconds. It then travels at constant speed in a straight line for 10 more seconds. At what point does the bee end up?

Question: What is speed and velocity?

Velocity is a vector: $\mathbf{v}(t) = \mathbf{r}'(\mathbf{t})$ Speed is a scalar: $|\,\mathbf{v}(t)\,|$

velocity: $\mathbf{v}(t) = \langle -2\sin(2t), 2\cos(2\,\mathrm{t}), 1\rangle$ Speed = $|\,\mathbf{v}(t)\,| = \sqrt{9} = 3$

At time $t_0 = 10,$ it travels along the tangent line with speed 3 ft/sec

tangent line: $\mathrm{s} \to \mathbf{r}(t_0) + s\mathbf{v}(t_0)$

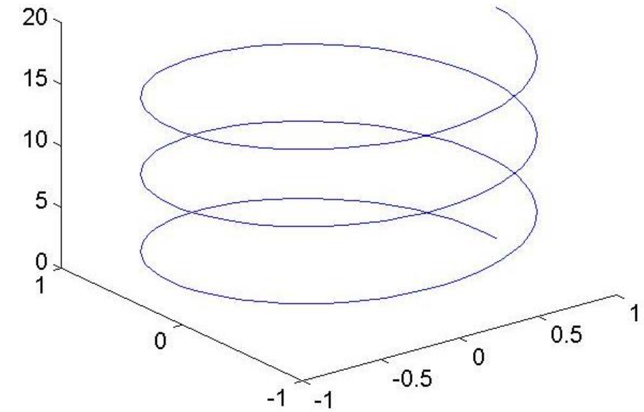
after 10 more seconds, it arrives at

$\mathbf{r}(t_0) + 10\mathbf{v}(t_0) =$

$< \cos(20), \sin(20), 10 > +10\langle -2\sin(20), 2\cos(20), 1\rangle$

$= \langle \cos(20) - 20\sin(20), \sin(20) + 20\cos(20), 20\rangle$

$\boxed{= \langle -17.85,\ 9.07,\ 20\rangle}$



Question: What is distance traveled?