Math 549 Spring 2010
Homework 1
Due: Thursday January 21 at the end of class.

Present your solutions clearly and legibly. If you are using results from Rudin’s book, you need to state them explicitly. An emphasis is placed on rigorous reasoning. A portion of the homework will be graded and returned to you.

(1) Construct a bounded set of real numbers with exactly three limit points.

(2) Let $E'$ be the set of limit points of a set $E$ in a metric space. Show that $E'$ is closed.

(3) Let $X$ be any infinite set and for $p, q \in X$ define the function

$$d(p, q) = \begin{cases} 
1 & \text{if } p \neq q, \\
0 & \text{if } p = q 
\end{cases}$$

Prove that this is a metric. Which subsets are open? closed? compact?

(4) If $x$ and $y$ are real numbers, define

$$d_1(x, y) = (x - y)^2; \quad d_2(x, y) = \sqrt{|x - y|}; \quad d_3(x, y) = |x^2 - y^2|;$$

$$d_4 = |x - 2y|; \quad d_5 = \frac{|x - y|}{1 + |x - y|}.$$ 

Which of these define metrics? Justify your assertions.

(5) Is the union (or intersection) of 2 compact sets compact? Is the union (or intersection) of an infinite number of compact sets compact?

(6) Let $K$ be a compact set in a metric space $X$ and let $p \in X - K$. Define $d(p, K) = \inf_{x \in K} d(p, x)$ as the distance between $p$ and $K$.

(a) Show there is at least one point $q \in K$ that has this minimum distance, so $d(p, q) = d(p, K)$

(b) Is there a unique such point $q$? Proof or counterexample.

(c) Is the assertion in part (a) still true if you only assume that $K$ is a closed (but not compact)? Proof or counterexample.

(7) Define two real numbers $x$ and $y$ to be equal if $|x - y|$ is an integer. Let $\alpha$ be an irrational real number, $0 < \alpha < 1$ and consider its integer multiples, $\alpha, 2\alpha, 3\alpha, \ldots$. Show that this set is dense in $0 \leq x \leq 1$. 