(1) Prove the following facts you know from calculus. Let \( f : (a, b) \to \mathbb{R} \) and \( x_0 \in (a, b) \) such that \( f \) is differentiable in \((a, b)\) and \( f''(x_0) \) exists.

(a) If \( f \) has a local min at \( x_0 \), then \( f'(x_0) = 0 \) (proved in class) and \( f''(x_0) \geq 0 \).

(b) If \( f'(x_0) = 0 \) and \( f''(x_0) > 0 \) show that \( f \) has a local min at \( x_0 \).

(2) If \( f \) has a (strict) local min at \( x_0 \) (i.e. \( f(x) > f(x_0) \) for \( x \neq x_0 \) but near \( x_0 \)), is it true that there exists a small neighborhood \((a, b)\) of \( x_0 \) such that \( f \) is decreasing on \((a, x_0)\) and increasing on \((x_0, b)\)? Prove or counter example.

(3) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^a \sin(x^b) \) for \( x \neq 0 \) and \( f(0) = 0 \) for some real numbers \( a, b \). Determine under which conditions on \( a, b \) is \( f \) is continuous.

When if \( f \) differentiable? When is \( f' \) continuous. More generally, for a given \( n \), when does \( f^{(n)}(x) \) exists (for all \( x \)) and when is it continuous.

(4) Let \( f : \mathbb{R} \to \mathbb{R} \) be a function with two continuous derivatives and \( f(0) = 2, \ f(1) = 0, \) and \( f(3) = 1 \). Prove there exists \( a \in (0, 3) \) with \( f''(a) > 2/3 \).

(5) Prove that a function \( f : \mathbb{R} \to \mathbb{R} \) with \( |f'(x)| < A \) for some \( A < 1 \) has a fixed point. Is this also true if \( A = 1 \)?

(6) Show that if \( f : E \to \mathbb{R}, \ E \subset \mathbb{R} \) is differentiable with \( |f'(x)| < A \) for some \( A > 0 \) and all \( x \in E \), then \( f \) is uniformly continuous.

(7) Prove the following version of L’Hopital: If \( \lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = \infty \) and \( \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L < \infty \), then \( \lim_{x \to \infty} \frac{f(x)}{g(x)} = L \) (assuming that \( g(x) \neq 0 \) and \( g'(x) \neq 0 \) for \( x \) large).

(8) (Extra Credit) Let \( f(x) = \alpha x + x^2 \sin(1/x) \) if \( x \neq 0 \) and \( f(0) = 0 \). If \( 0 < \alpha \leq 1 \), show that although \( f'(0) > 0 \), \( f \) is not increasing in any interval containing 0. But if \( \alpha > 1 \), show that \( f \) is increasing in some interval containing 0.