Math 549 Spring 2010
Homework 6
Due: Friday February 26 by 4pm in TA’s mailbox.

(1) Compute \( \int_a^b x^n \, dx \) directly from the definition of the integral.

(2) Define a function \( f : [0, 1] \to [0, 1] \) by \( f(x) = 0 \) if \( x \) irrational, and \( f(p/q) = 1/q \) for rational \( x = p/q \) in lowest terms. At what points is \( f \) continuous? At what points is \( f \) differentiable? Does \( \int_0^1 f \, dx \) exist?

(3) Define a function \( f : [0, 1] \to [0, 1] \) by \( f(x) = 0 \) if \( x \) irrational, and \( f(p/q) = 1 \) for rational \( x = p/q \) in lowest terms. Is \( f \) integrable?

(4) Let \( f : [a, b] \to \mathbb{R} \) be a non-negative continuous function. If \( \int_a^b f \, dx = 0 \), show that \( f = 0 \).

(5) Show that \( ||f|| := \left\{ \int_a^b |f|^2 \, dx \right\}^{1/2} \) is a norm on the vector space of continuous function defined on \([a, b]\).

(6) Show that \( \sin x \) is not a rational function. You can assume basic properties of \( \sin x \).

(7) (Extra Credit) Prove the Hölder inequality

\[
\left| \int_a^b f \, g \, dx \right| \leq \left\{ \int_a^b |f|^p \, dx \right\}^{1/p} \left\{ \int_a^b |g|^q \, dx \right\}^{1/q}
\]

for any positive real numbers \( p, q \) with \( \frac{1}{p} + \frac{1}{q} = 1 \), and \( f \) and \( g \) are integrable. Show that this implies that \( ||f||_p := \left\{ \int_a^b |f|^p \, dx \right\}^{1/p} \) is a norm.