Math 549 Spring 2010
Homework 7

Due: Friday March 5 by 4pm in TA’s mailbox.

(1) Formulate and prove a version of the substitution rule for integration.

(2) Let \( f \) be Riemann integrable on \([a, b]\).
   (a) Show that \( f \) is Riemann integrable on any subinterval.
   (b) Prove that \( \int_{a}^{b} f \, dx = \int_{a}^{c} f \, dx + \int_{c}^{b} f \, dx \) for any \( c \in [a, b] \)

(3) Prove a mean value theorem for integration: Let \( f \) be a continuous function on \([a, b]\). Show that there exists a value \( \alpha \in [a, b] \) such that \( \int_{a}^{b} f(x) \, dx = f(\alpha)(b - a) \).
    Give an example that this is not true if \( f \) is not continuous.

(4) Let \( f \) be continuous on \([0, \infty)\) with \( \lim_{x \to \infty} f(x) = c \). Show that
    \[
    \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} f(x) \, dx = c.
    \]

(5) Formulate and prove a version of the integral test for the convergence of an infinite series that you learned in calculus.

(6) Define \( F(x) = \int_{1}^{x} \frac{1}{t} \, dt \) for \( x > 0 \). Without using any knowledge about logarithms, show that \( F(xy) = F(x) + F(y) \).

(7) (Extra Credit) Find an example of 2 Riemann integrable functions whose composition is not Riemann integrable. Is the composition of a Riemann integrable function and a continuous function Riemann integrable?