Math 600 Fall 2017

Homework 1

Due: Thursday September 7 at the end of class. A portion of the homework will be graded (by Joe Hoisington) and returned to you.

(1) Show that $S^n$ is a $C^\infty$ manifold (all details) by:

(a) Using an atlas of two charts \{(S^n - N, \phi_N), (S^n - S, \phi_S)\} where $\phi_N$ (resp. $\phi_S$) represents stereographic projection from the north (resp. south) pole $N = (0,0,...,0,1)$ (resp. $S = (0,0,...,0,-1)$).

(b) Using the atlas \{(U^+_i, \phi^+_i) | i = 1, 2, ..., n + 1\} where $U^+_i = \{(x_1, x_2, ..., x_{n+1}) | \sum x_i^2 = 1, x_i > 0\}$ and $\phi^+_1(x_1, x_2, ..., x_{n+1}) = (x_1, x_2, ..., x_{i-1}, x_i+1, ..., x_{n+1})$.

(c) Show that these give rise to the same differentiable structure.

(2) Show that $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are manifolds and that they are compact.

(3) Show that the following three definitions of a continuous action by a discrete group $G$ on a manifold $M$ to be proper are equivalent:

(a) An action is proper if

(i) For each $p \in M$ there exists a neighborhood $U$ of $p$ such that $gU \cap U = \emptyset$ for all but finitely many $g$.

(ii) For any two points $p, q \in M$ not in the same $G$ orbit, there exist neighborhoods $U$ of $p$ and $V$ of $q$ such that the open sets $gU, g \in G$ and $g'V, g' \in G$ are all disjoint.

(b) An action is proper if for every compact set $K \subset M$ the set $\{g \in G | gK \cap K \neq \emptyset\}$ is finite.

(c) An action is proper if the map $M \times G \to M \times M: ((p, g) \to (p, gp))$ is continuous and proper (i.e. inverse images of compact sets are compact).

(4) Recall that an action is properly discontinuous if (ii) above holds and I replace (i) by

(i') For each $p \in M$ there exists a neighborhood $U$ of $p$ such that $gU, g \in G$ are all disjoint.

Show that a finite group acts properly discontinuously iff it acts freely.

(5) Show that if $G$ acts properly discontinuously and smoothly on a smooth manifold $M$, then $M/G$ is a smooth manifold such that the projection $M \to M/G$ is smooth.

(6) (Extra Credit): Show that $\mathbb{C}P^1$ is diffeomorphic to $S^2$. 

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