Due: Thursday October 1st at the end of class. A portion of the homework will be graded (by Martin Citoler-Saumell) and returned to you. Since you have 2 weeks, this homework is a little longer. If you have questions next week, you can ask Martin as well.

(1) Let \( f : \mathbb{RP}^2 \to \mathbb{R}^6 \) defined by
\[
[x, y, z] \mapsto (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz)
\]
(a) Show that \( f \) is an embedding of \( \mathbb{RP}^2 \) into \( \mathbb{R}^6 \) with image in \( S^5(1) \).
(b) If \( V^5 = \{ (y_1, \ldots, y_6) \in \mathbb{R}^6 \mid y_1^2 + y_2^2 + y_3^2 = 1 \} \) show that \( f(\mathbb{RP}^2) \subset V^5 \) and \( f(\mathbb{RP}^2) \subset V^5 \cap S^5(1) = S^4(\sqrt{2}/3) \). Hence \( f \) is an embedding into \( S^4(\sqrt{2}/3) \) which, via stereographic projection, induces an embedding into \( \mathbb{R}^4 \).

Remember that I mentioned that there is no embedding of \( \mathbb{RP}^2 \) into \( \mathbb{R}^3 \).

(2) Define a \( C^k \) derivation on a manifold for any finite integer \( k > 0 \), denoted by \( V_k \)
(a) Show that \( V_k \) is a vector space and define a natural injection of \( T_pM \) into this vector space.
(b) Explain why the proof that this injection is an isomorphism does not work any more as it did for \( C^1 \) derivations.
(c) Show that \( V_k \) is isomorphic to the dual space \( (F_p/F_p^2)^* \) where \( F_p \) is the vector space of functions that vanish at \( p \), and \( F_p^2 \) the linear subspace spanned by products of functions in \( F_p \).
(d) (Extra Credit) Show that \( V_k \) is infinite dimensional. (Try this first for functions from \( \mathbb{R} \) to \( \mathbb{R} \))

(3) Show that \( TM \) is an orientable manifold, no matter whether \( M \) is orientable or not.

(4) Let \( \Gamma \) be a group that acts properly discontinuously on an orientable manifold \( M \) with quotient \( M/\Gamma \). Recall we already showed that \( M/\Gamma \) is a manifold.
(a) Show that \( M/\Gamma \) is orientable iff the action of \( \Gamma \) consists of orientation preserving diffeomorphisms.
(b) Apply this to show that the Moebius band, the Klein bottle, and \( \mathbb{RP}^{2n} \) are not orientable and that \( \mathbb{RP}^{2n+1} \) and the lens spaces are orientable.

(5) Show that \([X,Y](f) = X(Y(f)) - Y(X(f))\) defines a vector field if \( X,Y \) is a vector field, and that it has the following properties:
(a) \([fX,Y] = f[X,Y] - Y(f)X \) and \([X,fY] = f[X,Y] + X(f)Y\).
(b) \([X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \) (the Jacobi identity).
(c) \([\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0 \) where \( x \) is a coordinate of \( M \).
(d) Find an expression for the Lie bracket in local coordinates (in terms of the components of \( X \) and \( Y \)).

(6) Compute the flow of the following vector fields:
(a) \( Z(a,b) = -b \frac{\partial}{\partial a} + a \frac{\partial}{\partial b} \) on \( \mathbb{R}^2 \).
(b) \( Z(a,b) = (4a - 3b) \frac{\partial}{\partial a} + (6a - 5b) \frac{\partial}{\partial b} \) on \( \mathbb{R}^2 \).
(c) \( Z(a) = (1 + a^2) \frac{\partial}{\partial a} \) on \( \mathbb{R} \).
Why are the vector fields in a) and b) complete (even before computing the flow)? Can you draw a "picture" of the flow?