Due: Friday September 22nd by 5pm, put it in Joe’s mailbox. A portion of the homework will be graded (by Joe Hoisington) and returned to you.

(1) Let \( f: \mathbb{RP}^2 \to \mathbb{R}^6 \) defined by
\[
[x, y, z] \mapsto (x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz)
\]

(a) Show that \( f \) is an embedding of \( \mathbb{RP}^2 \) into \( \mathbb{R}^6 \) with image in \( S^5(1) \).

(b) If \( V^5 = \{ (y^1, ..., y^6) \in \mathbb{R}^6 \mid y^1 + y^2 + y^3 = 1 \} \) show that \( f(\mathbb{RP}^2) \subset V^5 \) and \( f(\mathbb{RP}^2) \subset V^5 \cap S^5(1) = S^4(\sqrt{2}/3) \). Hence \( f \) is an embedding into \( S^4(\sqrt{2}/3) \) which, via stereographic projection, induces an embedding into \( \mathbb{R}^4 \).

Remember that I mentioned that there is no embedding of \( \mathbb{RP}^2 \) into \( \mathbb{R}^3 \).

(2) Define a \( C^k \) derivation on a manifold for any finite integer \( k > 0 \), denoted by \( V_k \)

(a) Show that \( V_k \) is a vector space and define a natural injection of \( T_pM \) into this vector space.

(b) Explain why the proof that this injection is an isomorphism does not work any more as it did for \( C^\infty \) derivations.

(c) Show that \( V_k \) is isomorphic to the dual space \( (F_p/F_p^2)^* \) where \( F_p \) is the vector space of functions that vanish at \( p \), and \( F_p^2 \) the linear subspace spanned by products of functions in \( F_p \).

(3) (Extra Credit) Show that \( V_k \) is infinite dimensional. (Try this first for functions from \( \mathbb{R} \) to \( \mathbb{R} \))