Due: Thursday October 8th at the end of class. A portion of the homework will be graded (by Martin Citoler-Saumell) and returned to you.

(1) Let \( f: M^n \to N^n \) be a diffeomorphism and \( X, Y \) are vector fields on \( M \) resp. \( N \).
   (a) Derive a formula for \( f_*(X) \) and \( f_*(Y) \) as derivations. Why does this show that these are differentiable vector fields?
   (b) Derive a formula for the flow of \( f_*(X) \) and \( f_*(Y) \) in terms of the flow of \( X \).

(2) Consider the Brieskorn variety 
\[ B_d = \{(z_0, \ldots, z_n) \in \mathbb{C}^n \mid z_0^d + z_1^2 + \cdots + z_n^2 = 0, \sum |z_i|^2 = 1\}. \]
   (a) Show that \( B \) is an embedded submanifold of \( \mathbb{C}^n \). What is its dimension?
   (b) Show that if \( d = 1 \) it is diffeomorphic to the sphere.
   (c) Show that if \( d = 2 \) it is diffeomorphic to the unit tangent bundle of the sphere: 
\[ T_1(S^n) = \{v \mid p \in S^n, v \in T_p(S^n) \subset \mathbb{R}^{n+1}, ||v|| = 1\} \]
   where the length of \( v \) is measured in the usual inner product on \( \mathbb{R}^{n+1} \).
   These examples are particularly interesting since it is known that for \( d, n \) odd and \( 2n - 1 = 1 \mod 8 \), \( B_d \) is homeomorphic but not diffeomorphic to a sphere (so called Kervaire spheres).

(3) Show that a \( k \)-dimensional distribution \( \Delta \) on \( M \) is integrable iff for every point \( p \in M \) there exist a neighborhood \( U \) of \( p \) and smooth vector fields \( X_1, \ldots, X_k \) such that \( X_1(q), \ldots, X_k(q) \) is a basis of \( \Delta_q \) for all \( q \in U \) and \([X_i, X_j] = \sum a^k_{ij} X_k \) for some smooth functions \( a_{ij} \).

(4) Let \( f: M \to N \) be a smooth map with only regular values. Show that \( \Delta_p := \{v \in T_p M \mid D(f)_p(v) = 0\} \) is an integrable distribution of \( M \). What are the maximal integral manifolds of \( \Delta \)?

(5) Let \( f: SL(2, \mathbb{R}) \to \mathbb{R} \) where \( f(A) = tr(A) \). Find the regular values of \( f \). What is the diffeomorphism type of the surface \( f^{-1}(q) \) for \( q \) regular? What does \( f^{-1}(q) \) look like for \( q \) singular?